LOCAL PROPERTIES OF ALGEBRAIC CORRESPONDENCES(1)

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Introduction. The author has developed in [1](2) a method of approach to the theory of algebraic correspondences between algebraic varieties; the method is based upon the consideration of a certain form $\Psi_{t,y\mathfrak{z}}$ associated to each cycle $\mathfrak{z}$ of an algebraic variety. Any logically sound method of defining algebraic systems of cycles on a variety V must display a one-to-one correspondence (strictly one-to-one!) between a variety G (that is, the set of its points) and the cycles of a set $\mathfrak{C}$ of cycles on V, the set $\mathfrak{C}$ being the one to be called an algebraic system of cycles. In the theory developed in [1] the variety G turns out to be constructed with the coefficients of the form $\Psi_{t,y}\Delta$, where Δ is the "general element" of $\mathfrak{C}$. The consideration of G is necessary in order to establish a 1-1 correspondence, and in order to prove that the cycles of V of a given order and dimension form an algebraic system (Theorem 5.5 of [1]); but Theorem 5.1 of [1] states that whenever an algebraic correspondence D between two varieties F, V has been established, the cycles of V which correspond to the points of F according to D form an algebraic system, with possibly the exception of the correspondents of the fundamental points of D on F, and of certain other points at which F has more than one sheet. Therefore the study of algebraic correspondences remains the basic tool for investigating properties of algebraic systems which are deeper than the mere foundations. Hence it is to be expected that the availability of various methods suited to the study of algebraic correspondences should prove to be useful in order to select, for any particular application, the one which yields results more readily or with more details. It is with this reason in mind that in the present paper, rather than selecting the shortest path to the main results, we indulge in giving detailed expressions for the "multiplicity" with which a given component has to be "counted" in constructing the cycle $D\{P\}$ of V corresponding to the point P of F according to D.

The manner of defining such multiplicity is the backbone of the whole theory of algebraic correspondences, since the actual determination of the components of $D\{P\}$ does not offer any difficulty. The definition used in [1] is the following one: those multiplicities are the exponents of the irreducible factors of the form $\Psi_{t,y}\Delta$, after operating a "specialization" on the coefficients

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of such form; however, Lemmas 5.3 and 5.4 of [1] give a hint to two other possible ways of defining such multiplicity, one of a local nature (or analytical), the other one, we might say, of a Galois-theoretical nature, and both equivalent to the original definition. These two possibilities are here exploited, and the results are stated respectively in Theorem 4.1 (or in the corollary to Theorem 5.1), and Theorem 5.2 (or 5.1). As a necessary step we also give a new short proof of the associativity formula for geometric local rings (see [3]).

We shall now proceed to give a brief description of the principal results of the present paper, and of their possible immediate applications. It is well known that the theory of intersection multiplicity of cycles of a variety and the theory of algebraic systems are intimately related to each other. It is possible to start from either end in the process of building the foundations of algebraic geometry. The first approach consists in (1) defining the algebraic systems of cycles in a manner independent of the intersection multiplicities and (2) using the algebraic systems in order to define the intersection multiplicities of two cycles, namely: if $\mathfrak{z}_1$, $\mathfrak{z}_2$ are two cycles of suitable (pure) dimensions in a projective space, consider two maximal algebraic systems $\mathfrak{C}_1$, $\mathfrak{C}_2$ containing $\mathfrak{z}_1$, $\mathfrak{z}_2$ respectively; assume, by definition, all the intersections of the general elements $\mathfrak{Z}_1$, $\mathfrak{Z}_2$ of $\mathfrak{C}_1$, $\mathfrak{C}_2$ to have the multiplicity 1 (at least in the case of characteristic zero); take the intersection $\mathfrak{Z}_1 \cap \mathfrak{Z}_2$ as the general cycle of a new algebraic system; "specialize" this cycle in such a way that $\mathfrak{Z}_i$ specializes to $\mathfrak{z}_i$. Then the multiplicities with which the various components with the right dimension appear in this specialization can be assumed, by definition, to be the multiplicities of intersection of $\mathfrak{z}_1$, $\mathfrak{z}_2$, *provided that such multiplicities depend only on* $\mathfrak{z}_1$, $\mathfrak{z}_2$.

The second approach consists in (1) defining the multiplicities of the intersections (having the right dimension) of two cycles in a way independent of the previous knowledge of algebraic systems, and (2) given the algebraic correspondence D between the varieties F, V, defining the cycle of V which corresponds to a point P of F to be the projection on V of the intersection $D \cap (V \times P)$, taken with the proper multiplicities. Now, Theorem 4.1 amounts to saying that *if the intersection multiplicities are defined according to* [3], *then the theory of algebraic correspondences as developed in* [1] *could be obtained by means of the second approach.* Algebraic geometers, however, have usually adopted the first approach, possibly because it requires less analytical means and more geometrical or topological ones(3); in doing so, it is necessary to show that the choice of $\mathfrak{C}_1$, $\mathfrak{C}_2$ does not influence the final result; when analyzed, this statement leads to the following problem: let, as before, D be an algebraic correspondence between F and V, and let W be an irreducible

(3) A more important reason is that the multiplicity of intersection has so far been defined only at a simple subvariety of the ambient variety, while the theory of correspondences does not have, and should not have, such limitations.

subvariety of V. By specializing the general point of V to the general point of W one obtains an algebraic correspondence D' between F and W, which we shall refer to as the *reduced* of D to (W, F) (this would correspond to the consideration of the algebraic system cut on W by the algebraic system of cycles on V generated by D, *if we had a theory of intersection multiplicities*); let now G be an irreducible subvariety of F, and denote by D'' the reduced of D' to (W, G); does D'' depend on the order in which the two reductions have been performed? The answer is *no* under ordinary circumstances; this is our "reduction theorem" (Theorem 4.2). Another difficulty, in the first approach, arises from the fact that $\mathfrak{z}_1 \cap \mathfrak{z}_2$ might have components of too high a dimension together with components of the right dimension. In other terms, the problem is the following one: if V, F, D have the previous meanings, let P be a point of F which is fundamental for D, but such that the variety (not the cycle) $D[P]$ which corresponds to P also has some component of the proper dimension; assume, for sake of simplicity, $D[P]$ to consist of two components, W_1 of the right dimension, and W_2 of a higher dimension; then the multiplicity with which W_1 should be counted is not defined directly, at least not in [1]; if, however, a point P' of F, not fundamental for D, "approaches" P, the cycle $D\{P'\}$ which corresponds to P' in D will have certain components, with certain multiplicities, approaching W_1, while the other components will approach subvarieties of W_2, which will depend on the manner in which P' approaches P. If the limit of the sum of the multiplicities of the components approaching W_1 does not depend on the manner in which P' approaches P, then such a limit could be taken as the multiplicity with which W_1 has to be counted. Now, Theorem 3.1 (or 5.3) states that this is actually the case, provided that F has exactly one sheet at P (this condition being sufficient but not necessary). This fact also provides an answer to a familiar problem in denumerative geometry, namely the problem of deciding for how many solutions an isolated solution of an algebraic system of equations "counts" when infinitely many other solutions are present.

The solution of the first problem, when stated only to the extent which is necessary in dealing with multiplicities of intersection, is contained in van der Waerden's work on algebraic geometry (see for instance [9]); the second problem is not solved there, and this is the reason why that intersection theory is incomplete. Both problems, when stated only to the extent which is necessary for intersection multiplicities, are implicitly solved in [3] for the very reason that the so-called uniqueness of the intersection multiplicity is there proved. After a suitable translation of the notations, one can recognize that Theorem 4, chap. III, §4 of [10] (stating the uniqueness of the multiplicity of specialization) is a special case of statement 4 of Theorem 3.1 of the present paper, or of statement 1 of Theorem 5.3, and of one of the remarks which precede Theorem 5.5. The criterion contained in Theorem 5, chap. III, §4 of [10], and its converse, are a particular case of the corollary to Theorem 5.6

of this paper. Theorem 5.6 itself would be called, in the language of [10], a *criterion for multiplicity* $[k(\bar{x}, \bar{y}):k(\bar{x})]_i$ (see chap. IX, §3 of [10]).

We shall finally remark that the methods used in section 3 are very similar to those of number theory (decomposition group of an ideal, and so on); they also implicitly provide a purely algebraic definition of the multiplicity of a geometric local ring, or, alternatively, link this multiplicity with the ramification properties of arbitrary valuations (not only discrete valuations of rank 1).

1. **Definitions and notations.** We shall adopt all the definitions and notations used in [1], with the exception of the modifications or generalizations which we shall state from time to time(4). We shall first repeat the definitions given in Theorem 2.2 and footnote 5 of [1].

Let V be an irreducible r-dimensional variety over the field k of characteristic p, and let $\{x_0, \cdots, x_n\}$ be its h.g.p. (homogeneous general point); set $y_i = \sum_{j=0}^{n} t_{ij}x_j$ $(i=0, \cdots, r+1)$, where the t's are indeterminates, and set $k^* = k(t)$. We have the following definitions:

$[k^*(x):k^*(y_0, \cdots, y_r)] = \text{ord } V =$ *order of* V;

$(k^*(x):k^*(y_0, \cdots, y_r)) = \text{red } V =$ *reduced order of* V;

$[k^*(y_0, \cdots, y_{r+1}):k^*(y_0, \cdots, y_r)] = \deg V =$ *degree of* V;

$\{k^*(x):k^*(y_0, \cdots, y_r)\} = \text{ins } V =$ *inseparability of* V;

$\{k^*(y_0, \cdots, y_{r+1}):k^*(y_0, \cdots, y_r)\} = \exp V =$ *exponent of inseparability of* V (this is a modification of the definition contained in footnote 5 of [1]);

$[k^*(x):k^*(y_0, \cdots, y_{r+1})] = h(V) =$ *strong inseparability of* V.

The last three numbers are powers of p if $p \neq 0$, and are equal to 1 if $p=0$.

The basic relations among these numbers are: ord $V = h(V)$ deg V, deg V = exp V red V, ins $V = h(V)$ exp V. If $e = \exp V$, we have that $L = k^*(y_0, \cdots, y_r, y_{r+1}^e)$ is the maximal subfield of $k^*(x)$ which is separable over $k^*(y_0, \cdots, y_r)$; e is also the degree (not the order) of $k^*(x)$ over L; if $e=1$, then necessarily $h(V) = \text{ins } V = 1$. A similar property holds true for deg V, that is, deg V is the degree of $k^*(x)$ over $k^*(y_0, \cdots, y_r)$. The proof of Theorem 2.2 of [1] implies that ins V is the maximum reached by $[k':k][k'(x):k(x)]^{-1}$ when k' ranges among the purely inseparable finite extensions of k, and this fact proves that ins V is a birational invariant of V, that is, it depends only on $k(V)$; we shall speak, therefore, of the *inseparability of* $k(V)$ *over* k, in symbols ins $(k(V):k) = \text{ins } V$. The proof of Theorem 2.2 of [1] also shows that exp V is the smallest integer e for which there exists a purely inseparable finite extension k' of k, of degree e over k, such that $[k':k][k'(x):k(x)]^{-1} = \text{ins } V$; as a consequence, exp V is also a birational invariant, and as such it will be denoted by exp $(k(V):k)$. The relation $h(V)$ exp $V = \text{ins } V$ implies then that $h(V)$ is a birational invariant, to be denoted by $h(k(V):k)$. On the other hand, it is well known that red V,

(4) The correct definition of $\{L:H\}$ (p. 455 of [1]) is $\{L:H\} = [L:L']$; clearly the alternative definition offered in [1], namely $\{L:H\} = [L'':H]$, is generally incorrect when L' (or L) is not normal over H.

deg V, ord V are not birational invariants. We shall also say that they are respectively the *reduced order*, the *degree*, and the *order of* $k(V)$ *over* k *with respect to* $\{x\}$, and shall denote them by red $(k(V):k;\, x)$, deg $(k(V):k;\, x)$, ord $(k(V):k;\, x)$. The three birational invariants clearly have a meaning also when V is an irreducible pseudovariety. Finally, by requesting that red, deg, ord be additive operators, they can be defined for any pure dimensional reducible variety or for any unmixed cycle; when this is done, red V manifests a property of invariance for extensions of the ground field, namely: if K is any extension of k, then red $V_K =$ red V. If ins $V=1$, then no element of $k(V)$ is purely inseparable over k, but the converse is not true.

Let again V be an irreducible variety over the field k, and let $\mathfrak{z}$ be the cycle $1\,V$; if K is a finite extension of k, the extension V_K of V over K has been defined in section 1 of [1]; nothing has been said of the extension $\mathfrak{z}_K$ of $\mathfrak{z}$ over K. Clearly any definition of $\mathfrak{z}_K$, in order to be useful, must satisfy the relation rad $\mathfrak{z}_K = V_K$; if K is separable over k, this is easily accomplished by defining $\mathfrak{z}_K = 1\,V_1 + \cdots + 1\,V_n$, if $V_1, \cdots, V_n$ are the components of V_K; as a consequence of this definition we have $\Psi_{t,y}\mathfrak{z}_K = \Psi_{t,y}\mathfrak{z}$, deg $\mathfrak{z}_K =$ deg $\mathfrak{z}$, red $\mathfrak{z}_K =$ red $\mathfrak{z}$, ord $\mathfrak{z}_K =$ ord $\mathfrak{z}$. If K is purely inseparable over k, no definition of $\mathfrak{z}_K$ will be such that these four relations are fulfilled in the most general case, and we can choose a definition of $\mathfrak{z}_K$ in such a way that $\mathfrak{z}$ and $\mathfrak{z}_K$ have either (1) the same order, or (2) the same degree, or (3) the same reduced order. This means that $\mathfrak{z}_K = r\,V_K$, where r is given, in cases (1), (2), (3), respectively by (ins V) $\times$(ins $V_i)^{-1}$ or (exp V)(exp $V_i)^{-1}$ or 1. We also have:

Case (1) $h(V_i)\Psi_{t,y}\mathfrak{z}_K = h(V)\Psi_{t,y}\mathfrak{z}$;
Case (2) $\Psi_{t,y}\mathfrak{z}_K = \Psi_{t,y}\mathfrak{z}$;
Case (3) exp $V\Psi_{t,y}\mathfrak{z}_K =$ exp $V_i\Psi_{t,y}\mathfrak{z}$.

We adopt here the definition of case (2), although we shall see that local properties put the emphasis rather on the order than on the degree; this discrepancy becomes immaterial when dealing with algebraic systems, since they are sets of cycles over an algebraically closed ground field. Case (2) gives a definition for any extension K over k, algebraic or transcendental, separable or inseparable, and remains unaltered when $\mathfrak{z}$ is any unmixed cycle.

If $\mathfrak{z}$ is an unmixed cycle over k, a field k' is said to be a *field of definition of* $\mathfrak{z}$ if there exist a field K containing k' and k as subfields, and a cycle $\mathfrak{z}'$ over k', such that $\mathfrak{z}_K = \mathfrak{z}'_K$.

Let F, V be two irreducible varieties over the field k, Δ an unmixed algebraic correspondence between $k(F)$ and V, and set $D = D_{\Delta,F}$; if v, P are a place and a point of F, the symbols $\Delta\{v\}$, $D\{P\}$ have been defined in [1] when k is algebraically closed. We wish to extend the meaning of these symbols to the case in which k is not necessarily algebraically closed, and v, P are any valuation of $k(F)$ and any irreducible subvariety of F.

Case 1: k is not necessarily algebraically closed, and $v \in M(F)$. Set $k' = K_v$, and denote by $\{c_0, \cdots, c_n\}$ the h.g.p. of G_Δ, so that the c's are proportional to the coefficients of the polynomial $\Psi_{t,y}\Delta \in k(F)[t, y]$. Let, say, c_0 be such that $c_i c_0^{-1} \in R_v$, and let ϕ be the polynomial obtained by replacing, in $\Psi_{t,y}\Delta$, c_0 by 1 and c_i $(i>0)$ by the residue class of $c_i c_0^{-1}$ (mod $\mathfrak{P}_v$), which is an element of k'. We contend that there exists a cycle $\mathfrak{z}$ of $V_{k'}$ such that $\Psi_{t,y}\mathfrak{z} = \phi$. In fact, let $\bar{k}$ be the algebraic closure of k', and let F' be any component of $F_{\bar{k}}$; call v' any extension of v to $\bar{k}(F')$; let the cycle Δ' be the extension of the cycle Δ over $\bar{k}(F')$, so that $\Psi_{t,y}\Delta' = \Psi_{t,y}\Delta$. Now, Δ' is an algebraic correspondence between $\bar{k}(F')$ and $V_{\bar{k}}$, and $\mathfrak{z}' = \Delta'\{v'\}$, which is defined because $\bar{k}$ is algebraically closed, is a cycle of $V_{\bar{k}}$ having k' as a field of definition; hence $\mathfrak{z}'$ is the extension over $\bar{k}$ of a cycle $\mathfrak{z}$ of $V_{k'}$, and $\mathfrak{z}$ fulfils the condition $\Psi_{t,y}\mathfrak{z} = \Psi_{t,y}\mathfrak{z}' = \phi$, which is what we wanted to prove. The cycle $\mathfrak{z}$ will be denoted by $\Delta\{v\}$.

Case 2: v is any valuation of $k(F)$ over k (and k is arbitrary). Let $\{z\}$ be the h.g.p. of F, and let H be a subfield of R_v, purely transcendental over k, of transcendency equal to dim v/k. Let F' be the irreducible variety over H whose h.g.p. is $\{z\}$, and set $V' = V_H$, so that V' is irreducible. Δ is an unmixed algebraic correspondence Δ' between $H(F')$ and V', and we have $\Psi_{t,y}\Delta' = \Psi_{t,y}\Delta$; however, $G_{\Delta'}$ is a variety over H. Now v is a place of F', and $\Delta'\{v\}$ is defined by case 1 and is a cycle of $V'_{K_v} = V_{K_v}$; clearly $\Delta'\{v\}$ does not depend on the choice of H. We shall define $\Delta\{v\} = \Delta'\{v\}$; $\Delta\{v\}$ is an unmixed algebraic correspondence between K_v and V, and has the property: if u is a valuation of K_v over k, and w is the valuation of $k(F)$ compounded of v and u, then $\Delta\{w\} = (\Delta\{v\})\{u\}$.

Case 3: let G be an irreducible subvariety of F such that there exists an i, say $i=0$, for which $c_j c_0^{-1} \in Q(G/F)$ for any j. Let v be any valuation of $k(F)$ whose center on F is G. Since $K = k(G) \subseteq K_v$, $\Delta\{v\}$ is the extension over K_v of a cycle of V_K. Such cycle will be denoted by $D\{G\}$, and does not depend on v. Finally, we shall define $\{D; V, G\} = \{D; G, V\}$ to be $D_{D\{G\},G}$. Notice that $D\{F\}$ exists and equals $\Delta = \Delta_{D,V}$, and that $\{D; V, F\} = D$.

Since $Q(P/F) \subseteq Q(G/F)$ for any $P \in G$, we see that $D\{G\}$ certainly exists if $D\{P\}$ exists for at least one $P \in G$; the converse is not true. If $P \in G$ and $\{D; V, P\}$ exists, then $\{\{D; V, G\}; V, P\}$ exists and equals $\{D; V, P\}$; however, $\{\{D; V, G\}; V, P\}$ may exist even when $\{D; V, P\}$ does not exist. Lemma 3.1 of [1] provides, in a crude way, a correspondence D' between G and V induced by D; namely, $D' = \text{rad } D'$ and $\wp(D'/G \times V) = \text{rad } \sigma\wp(\text{rad } D/F \times V)$, where σ is the homomorphic mapping of $k[x, z]$ with kernel $\wp(G/F)k[x, z]$. The link between D' and $\{D; V, G\}$ (when this exists) is the following one: the components of $\{D; V, G\}$ are all and only those components of D' which operate on the whole G.

In order to take care of the outstanding role played by the order (instead of the degree) in the local properties of algebraic correspondences, we shall give the following definitions: let D be an irreducible algebraic correspondence between the irreducible varieties F, V over k, operating on the whole F; let G

be an irreducible subvariety of F such that $\{D;\ V,\ G\}$ exists, and let us assume $\{D;\ V,\ G\}=\sum_i\alpha_iD_i$, the D_i's being distinct irreducible pseudovarieties. As a consequence, we also have $D\{G\}=\sum_i\alpha_iD_i\{G\}$; we shall define $\{D;\ V,\ G\}^*=\sum_i\alpha_i^*D_i$, $D\{G\}^*=\sum_i\alpha_i^*D_i\{G\}$, where $\alpha_i^*=\alpha_ih(D\{F\})(h(D_i\{G\}))^{-1}$. Notice that $D\{F\}^*=D\{F\}$. Similarly, if $\Delta=D\{F\}$ and $\Delta\{v\}=\sum_i\beta_i\Delta_i$, we define $\Delta\{v\}^*=\sum_i\beta_i^*\Delta_i$, where $\beta_i^*=\beta_ih(\Delta)/h(\Delta_i)$. If D is not irreducible but is unmixed, the extension of these definitions is self-evident. We now have: ord $\Delta\{v\}^*=$ord $D\{G\}^*=$ord $\Delta=$ord $D\{F\}$, which replace (7) and (9) of [1]. As a consequence of Theorems 3.2 and 3.5 of [1] we also have: dim $D\{G\}=$dim $D-$dim F, dim $\{D;\ V,\ G\}=$dim $D-$dim $F+$dim G, dim $\Delta\{v\}=$dim $\Delta=$dim $D-$dim F. The number α_i^* is not necessarily an integer; we defer until section 5 the discussion of whether certain numbers which we shall obtain from time to time are integers or not. Here we remark only that $\{D;\ V,\ G\}^*$ is a *rational* (effective) *cycle*, that is, a formal linear combination of irreducible pseudovarieties with rational (positive) coefficients. Most notations and definitions valid for cycles can be easily extended to rational cycles; from now on, *cycle* shall mean rational (effective) cycle, and we shall use the expression *integral cycle* to denote a cycle with integral coefficients.

If D is irreducible, then $D'=$rad $\{D;\ V,\ G\}$ is a pseudosubvariety of D or also of $G\times V$; if $\{x,\ \zeta\}$ is a general point of D, homogeneous in the set $\{x\}$, and such that $\{\zeta\}$ is a n.h.g.p. of F for which G is at finite distance, then the minimal primes of $\wp(D'/k[x,\ \zeta])$ are given by $\mathfrak{P}\cap k[x,\ \zeta]$ when $\mathfrak{P}$ ranges among all the minimal primes of $\mathfrak{P}(G/F)Q(G/F)[x]$; the components of $\Delta\{v\}$ are likewise in 1-1 correspondence with the minimal primes of $\mathfrak{P}_vR_v[x]$.

When $\Delta\{v\}$ or $\{D;\ V,\ G\}$ do not exist, we may still define $\Delta(v)$, $\Delta[v]$, $D(G)$, $D[G]$, $(D;\ V,\ G)$, $[D;\ V,\ G]$ in the following manner:

$\Delta'=\Delta(v)$ if Δ' is an algebraic correspondence between K_v and V such that any n.h.g.p. of Δ' is obtained by reducing mod $\mathfrak{P}_w$ a n.h.g.p. of Δ, where w is a valuation of $k(F)(\Delta)$ which induces v in $k(F)$;

$\Delta[v]$ is the join of all the $\Delta(v)$; with the same proof as the one of Theorem 3.2 of [1] we derive that if $\{x\}$ is the h.g.p. of Δ, then the components of $\Delta[v]$ are obtained in the following way: consider any minimal prime $\mathfrak{P}$ of $\mathfrak{P}_vR_v[x]$; then the cosets of the x's mod $\mathfrak{P}$ give the h.g.p. of a component of $\Delta[v]$, and conversely. We have $\Delta[v]=$rad $\Delta\{v\}$ if the latter exists;

$(D;\ V,\ G)$ is any subvariety D' of D such that $\wp(D'/D)\cap k[z]=\wp(G/F)$, $\{z\}$ being the h.g.p. of F;

$[D;\ V,\ G]$ is the join of all the $(D;\ V,\ G)$, and again a result similar to Theorem 3.5 of [1] holds true; we have $[D;\ V,\ G]=$rad $\{D;\ V,\ G\}$ if the latter exists;

$D(G)=\Delta_{V,(D;V,G)}$;

$D[G]=\Delta_{V,[D;V,G]}$; we have $D[G]=$rad $D\{G\}$ if the latter exists;

Finally, the *total transform* $\{D;\ V,\ G]$ *of* G is the join of all the $(D;\ V,\ P)$

for $P \in G$; we shall never have occasion to use $\{D; V, G]$ in this paper.

Notice that the symbol $D[P]$, as used in [1], coincides with the present definition of $D[P]$ only if P is a rational point of F.

2. **The multiplicity of the quotient ring of a variety.** We shall deal with those particular local rings which are quotient rings of an irreducible subvariety of an irreducible variety over a field; these rings will be called *geometric domains*; we say that a local ring is a *local domain* if it is an integral domain. The expression "geometrical local ring" of [3] has a more general meaning than our geometric domains. We assume the reader to be familiar with the theory of local rings contained in [2].

If R is a local ring with the maximal prime $\mathfrak{m}$, a subfield K of R, such that $R/\mathfrak{m}$ is a finite extension of the field K' consisting of the cosets of the elements of K mod $\mathfrak{m}$, will be called a *basic field of* R, and will be identified with K'; geometric domains and their completions contain basic fields. Other properties of geometric domains of frequent use are the following ones (see [12] for the proofs):

A geometric domain is *analytically unramified*, that is, its completion is semi-simple (its radical is the zero ideal);

Any prime ideal $\mathfrak{p}$ of a geometric domain R is *analytically unramified*, that is, $\mathfrak{p}R^* = \text{rad } \mathfrak{p}R^*$ if R^* is the completion of R; this is a consequence of the fact that $R/\mathfrak{p}$ is also a geometric domain;

A geometric domain is *analytically unmixed*, that is, each minimal prime of the zero ideal of its completion has the same dimension;

Any prime ideal $\mathfrak{p}$ of a geometric domain R is *analytically unmixed*, that is, each minimal prime of $\mathfrak{p}R^*$ has the same dimension, R^* being the completion of R;

An integrally closed geometric domain is *analytically irreducible*, that is, its completion is an integral domain.

We say that the irreducible variety V over k is *analytically irreducible at* its irreducible subvariety W if $Q(W/V)$ is analytically irreducible.

If R is an integral domain, $\mathfrak{K}(R)$ shall denote its quotient field.

If R is a local ring, $\{\zeta_1, \cdots, \zeta_r\}$ a set of nonunits of R, K a subfield of R, we shall denote by $K\{\zeta_1, \cdots, \zeta_r]$ or $K\{\zeta]$ the ring consisting of the limits, in the completion of R, of the sequences $\{f_n\}$, where $f_n = \sum_{i=0}^{n} \phi_i$, ϕ_i being a form of degree i in $\zeta_1, \cdots, \zeta_r$ with coefficients in K; if $K\{\zeta]$ is an integral domain, then we shall put $K\{\zeta\} = K\{\zeta_1, \cdots, \zeta_r\} = \mathfrak{K}(K\{\zeta])$.

We shall also make use of the results of [12] and [13], but we do not assume any knowledge of the results of [3].

Let R be a complete local ring with the maximal prime $\mathfrak{m}$, and assume it to be *unmixed*, that is, such that $R/\mathfrak{r}_i$ have all the same dimension when $\mathfrak{r}_i$ $(i=1, \cdots, n)$ ranges among the minimal primes of the zero ideal; let K be any basic field of R (R being such that it has a basic field), and let $\{\zeta_1, \cdots, \zeta_r\}$ be a set of parameters of R; let S be the multiplicatively closed set of the nonzero elements of $K\{\zeta\}$; then no element of S is a divisor of zero in R, so that R_S exists; according to results of [2], R_S is an algebra of finite order over $K\{\zeta\}_S = K\{\zeta\}$; the number $[R_S:K\{\zeta\}]\,[R/\mathfrak{m}:K]^{-1}$ is an integer independent of K, and is called the *multiplicity of R for $\{\zeta\}$* and denoted by $e(R;\zeta) = e(R;\zeta_1, \cdots, \zeta_r)$. If R is a geometric domain, $\{\zeta\}$ a set of parameters of R, R^* the completion of R, then by definition the *multiplicity of R for $\{\zeta\}$*, denoted by $e(R;\zeta) = e(R;\zeta_1, \cdots, \zeta_r)$, is given by $e(R^*;\zeta)$.

The following result is contained in [2] (more exactly is contained in the proofs of the lemmas which prepare the definition of multiplicity):

LEMMA 2.1. *Let R be a complete unmixed local ring, $\{\zeta_1, \cdots, \zeta_r\}$ a set of parameters of R; let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ be the minimal primes of $\zeta_1 R$, and let $\mathfrak{q}_i$ be the isolated primary component of $\zeta_1 R$ belonging to $\mathfrak{p}_i$; set $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$, and denote by σ, σ_i, τ_i the homomorphic mappings of R having respectively the kernels $\mathfrak{a}$, $\mathfrak{q}_i$, $\mathfrak{p}_i$; let l_i be the length of $\mathfrak{q}_i$. Then $\{\sigma\zeta_2, \cdots, \sigma\zeta_r\}$, $\{\sigma_i\zeta_2, \cdots, \sigma_i\zeta_r\}$, $\{\tau_i\zeta_2, \cdots, \tau_i\zeta_r\}$ are sets of parameters respectively of σR, $\sigma_i R$, $\tau_i R$, and we have: $e(R;\zeta_1, \cdots, \zeta_r) = e(\sigma R;\sigma\zeta_2, \cdots, \sigma\zeta_r) = \sum_{i=1}^{s} e(\sigma_i R;\sigma_i\zeta_2, \cdots, \sigma_i\zeta_r) = \sum_{i=1}^{s} l_i e(\tau_i R;\tau_i\zeta_2, \cdots, \tau_i\zeta_r)$.*

In the course of the next few proofs it is convenient to use some topological means; although the notations used here are the usual ones, we shall give a short description of them.

If R is a ring, a *metric in R* is a sequence $\{\mathfrak{A}_i\}$ $(i=1, 2, \cdots)$ of ideals of R such that $\mathfrak{A}_{i+1} \subseteq \mathfrak{A}_i$ and that $\bigcap_{i=1}^{\infty} \mathfrak{A}_i = 0$. Two metrics $\{\mathfrak{A}_i^{(1)}\}$, $\{\mathfrak{A}_i^{(2)}\}$ are *equivalent* if for each integer i there exist two integers $j(i)$ and $h(i)$ such that $\mathfrak{A}_{j(i)}^{(1)} \subseteq \mathfrak{A}_i^{(2)}$ and $\mathfrak{A}_{h(i)}^{(2)} \subseteq \mathfrak{A}_i^{(1)}$. A *topology in R* is the set of all the metrics equivalent to a given one; this amounts to saying that a topology in R is defined by giving a metric $\{\mathfrak{A}_i\}$, and by defining the neighborhoods of an element $a \in R$ to be the cosets of a (mod $\mathfrak{A}_i$). If T is a topology in R and $\{\mathfrak{A}_i\}$ is any metric belonging to T, then the property of a sequence of elements of R of being a Cauchy-sequence or a zero-sequence according to the metric $\{\mathfrak{A}_i\}$ depends only on T, so that we shall speak of *T-Cauchy-sequences* or *T-zero-sequences*. Then the meaning of *T-completion of R* is clear; the T-completion of R will be consistently denoted by R_T; it is well known that operations in R_T can be defined in such a way that R_T becomes a ring containing R as a subring and a subspace. If T_1, T_2 are topologies of the ring R, we shall write $T_1 \subset T_2$ if $T_1 \neq T_2$ and if in addition every T_1-zero-sequence is a T_2-zero-sequence. Since a sequence $\{a_i\}$ is a T_1-Cauchy-sequence if and only if the sequence $\{b_i\}$, where $b_i = a_{i+1} - a_i$, is a T_1-zero-sequence, we see that the notation $T_1 \subset T_2$ is justified. If T_i $(i=1, 2)$ contains the metric $\{\mathfrak{A}_j^{(i)}\}$, then the metric $\{\mathfrak{A}_j^{(1)} \cap \mathfrak{A}_j^{(2)}\}$ belongs to a topology T which depends only on T_1 and T_2; we shall set $T = T_1 \cap T_2$. A sequence is a T-zero-sequence if and only if it is a T_i-zero-sequence for $i=1, 2$, so that $T_1 \cap T_2$ is the "largest" topology T such that $T \subseteq T_i$ $(i=1, 2)$.

If R is a local or semi-local ring in the sense of [2], the *R-topology* shall be the natural topology of R.

Let T, T' be topologies of a ring R, such that $T\subseteq T'$, and denote by T also the natural extension of T in R_T; then $R_{T'}$ contains a homomorphic image of R_T, and the related homomorphic mapping is continuous; the kernel of such a mapping will be denoted by $\mathfrak{K}(T'/T)$, and it is the set of the T-limits of those T-Cauchy-sequences of elements of R which are T'-zero-sequences. A result of [2] states that if R is a complete semi-local ring and T' is the R-topology, then $T\subseteq T'$ for any topology T of R; and another result of [2] states that if T' is the R-topology of a semi-local ring R, and $T\subseteq T'$ is a topology of R induced in R by the S-topology of some semi-local ring S containing R and such that each maximal prime of S contains the intersection of the maximal primes of R, then the homomorphic mapping whose kernel is $\mathfrak{K}(T'/T)$ is a mapping of R_T *onto* $R_{T'}$. These facts will be freely used without making particular mention of them.

LEMMA 2.2. *Let R be a geometric domain, K a finite extension of $H=\mathfrak{K}(R)$, R' a subring of K containing R, integrally dependent on R, and such that $K=\mathfrak{K}(R')$. If $\mathfrak{m}$ is the maximal prime of R, let $\mathfrak{m}_1', \cdots, \mathfrak{m}_r'$ be the distinct primes of R' which lie on $\mathfrak{m}$, and set $R_i=R'_{\mathfrak{m}_i'}$, $\mathfrak{m}_i=\mathfrak{m}_i'R_i$. Let $\{\zeta\}$ be a set of parameters of R, so that $\{\zeta\}$ is also a set of parameters of each R_i; then the following equality holds true: $\sum_{i=1}^{r}[R_i/\mathfrak{m}_i:R/\mathfrak{m}]e(R_i;\zeta)=[K:H]e(R;\zeta)$.*

Proof. Case 1. Assume R and R' to be integrally closed, so that R' is the integral closure of R in K, and is a semi-local ring in the sense of [2]. Let T, T', T_i be the R-topology, the R'-topology, and the R_i-topology respectively. Then T_i and T' induce T in R by Theorem 3 of [13], so that $R_i^*=(R_i)_{T_i}$ and $R'^*=R'_{T'}$ both contain $R^*=R_T$; R_i^* and R^* are integral domains. From the definition of multiplicity we obtain

$$(1)\qquad [R_i/\mathfrak{m}_i:R/\mathfrak{m}]e(R_i;\zeta) = [\mathfrak{K}(R_i^*):\mathfrak{K}(R^*)]e(R;\zeta).$$

Let $b_i\in R'$ $(i=1,\cdots,\mu)$ be such that $\{b_i\}$ is an H-independent basis of K; since R' is a finite R-module, there exists an element $b\in R$ such that every element of bR' can be expressed in exactly one way as a linear combination of $b_1,\cdots,b_\mu$ with coefficients in R. Let $a\in R'^*$, and suppose $a=T'$-lim a_j, $a_j\in R'$, so that $ba=T'$-lim ba_j; set $ba_j=\sum_i a_{ji}b_i$, $a_{ji}\in R$. Since $\{a_j\}$ is a T'-Cauchy-sequence, we have

$$\sum_i (a_{ji}-a_{j-1,i})b_i \in b\mathfrak{m}^{n(j)}R',$$

where $n(j)$ approaches infinity with j; hence $a_{ji}-a_{j-1,i}\in\mathfrak{m}^{n(j)}$, that is, $\alpha_i=T\text{-lim}_{j\to\infty}a_{ji}$ exists, so that $ba=\sum_i\alpha_ib_i$, or $a=\sum_i b^{-1}\alpha_ib_i$. We have thus proved that $\{b_i\}$ is a $\mathfrak{K}(R^*)$-basis for R'^*_S, where S denotes the multiplicatively closed set consisting of the nonzero elements of R^*. It is also a $\mathfrak{K}(R^*)$-

independent basis since the previous argument shows that if a vanishes then each α_i has to vanish. But we have $R_S'^* = (R_1^*)_S \dotplus \cdots \dotplus (R_r^*)_S$ by a result of [2], so that $(R_i^*)_S$ is a commutative algebra of finite order over $\mathfrak{K}(R^*)$, and has no divisor of zero; hence $(R_i^*)_S = \mathfrak{K}(R_i^*)$, and we have proved that $\sum_i [\mathfrak{K}(R_i^*):\mathfrak{K}(R^*)] = [K:H]$. This and formula (1) yield the contention in the present case.

Case 2. Assume $K=H$, and take for R' the integral closure of R. Let T, T', T_i, R^*, R'^*, R_i^* have the same meanings as in case 1. Denote also by T_i', T_i the topologies induced respectively in R', R by T_i; then $T' = T_1' \cap \cdots \cap T_r'$, $T = T_1 \cap \cdots \cap T_r$, and $R'^* = R_1^* \dotplus \cdots \dotplus R_r^*$ (remember that $R_i^* = R'_{T_i'}$). Since R' is a finite R-module, there exists an element $b \in R$ such that $bR' \subseteq R$, hence $bR_i^* \subseteq R_{T_i}$, and also $\mathfrak{K}(R_i^*) = \mathfrak{K}(R_{T_i})$. As a consequence we have

$$e(R_i; \zeta) = [\mathfrak{K}(R_{T_i}) : k\{\zeta\}][R_i/\mathfrak{m}_i : k]^{-1},$$

k being a basic field of R. Set now $\mathfrak{r}_i = \mathfrak{R}(T_i/T)$, so that $R_{T_i} \cong R^*/\mathfrak{r}_i$. From the relation $T_1 \cap \cdots \cap T_r = T$ we derive $\mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_r = (0)$; no $\mathfrak{r}_i$ is zero, and no $\mathfrak{r}_i$ contains another $\mathfrak{r}_j$ for $j \neq i$, since $\mathfrak{r}_i \subseteq \sum_{j \neq i} R_j^*$. Hence the $\mathfrak{r}_i$'s are all the distinct minimal primes of zero, so that, by the definition of multiplicity, we have $e(R;\zeta) = \sum_i e(R^*/\mathfrak{r}_i; \zeta) = \sum_i [\mathfrak{K}(R_{T_i}) : k\{\zeta\}][R/\mathfrak{m} : k]^{-1}$; this, and the previous result, yield our contention.

Case 3 (general case). In this case we set S = integral closure of R in H, S' = integral closure of R in K, and denote by $\mathfrak{p}_1, \mathfrak{p}_2, \cdots$ the minimal primes of $\mathfrak{m}S$, and by $\mathfrak{p}_{j1}', \mathfrak{p}_{j2}', \cdots$ the minimal primes of $\mathfrak{m}_j' S'$; we also set $S_i = S_{\mathfrak{p}_i}$, $S_{ij}' = S'_{\mathfrak{p}_{ij}'}$. Then, by case 1, $\sum^{(l)} [S_{ij}'/\mathfrak{p}_{ij}' S_{ij}' : S_l/\mathfrak{p}_l S_l] e(S_{ij}'; \zeta) = [K:H] e(S_l; \zeta)$, where $\sum^{(l)}$ ranges over all the values of i, j such that $\mathfrak{p}_l \subseteq \mathfrak{p}_{ij}'$. By case 2 $\sum_l [S_l/\mathfrak{p}_l S_l : R/\mathfrak{m}] e(S_l; \zeta) = e(R; \zeta)$, so that $\sum_{ij} [S_{ij}'/\mathfrak{p}_{ij}' S_{ij}' : R/\mathfrak{m}] e(S_{ij}'; \zeta) = [K:H] e(R; \zeta)$. Now, by case 2 we have $\sum_j [S_{ij}'/\mathfrak{p}_{ij}' S_{ij}' : R_i/\mathfrak{m}_i] e(S_{ij}'; \zeta) = e(R_i; \zeta)$, which, with the previous formula, gives our contention, Q.E.D.

LEMMA 2.3. *Let R be a geometric domain; let $\{\zeta_1, \cdots, \zeta_r\}$, $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$, $\tau_1, \cdots, \tau_s$ have the same meanings as in Lemma* 2.1. *Let $v_{i1} v_{i2}, \cdots$ be all the distinct normalized*(5) *nontrivial discrete valuations of rank* 1 *of $\mathfrak{K}(R)$ such that $\mathfrak{C}(v_{ij}/R) = \mathfrak{p}_i$, and set $l_i = \sum_j v_{ij}(\zeta_1) [K_{v_{ij}} : \mathfrak{K}(R/\mathfrak{p}_i)]$. Then*

$$e(R; \zeta_1, \cdots, \zeta_r) = \sum_i l_i e(\tau_i R; \tau_i \zeta_2, \cdots, \tau_i \zeta_r).$$

Proof. Case 1. Assume R to be integrally closed, and let R^* be the completion of R, so that R^* is an integral domain. By definition, we have $e(R;\zeta) = e(R^*; \zeta)$. Let $\mathfrak{p}_{i1}, \mathfrak{p}_{i2}, \cdots$ be the minimal primes of $\mathfrak{p}_i R^*$, and call $\mathfrak{q}_{ij}$ the isolated primary component of $\zeta_1 R^*$ belonging to $\mathfrak{p}_{ij}$, and l_{ij} its length. If τ_{ij} is the homomorphic mapping of R^* whose kernel is $\mathfrak{p}_{ij}$, Lemma 2.1 implies

(5) A nontrivial discrete valuation of rank 1 is said to be *normalized* if its value-group is the additive group of integers.

$e(R^*; \zeta) = \sum_{ij} l_{ij} e(\tau_{ij}R^*; \tau_{ij}\zeta_2, \cdots, \tau_{ij}\zeta_r)$. Now, by Lemma 3 of [12], there exists exactly one nontrivial normalized discrete valuation w_{ij} of rank 1 of $\mathcal{K}(R^*)$ whose center on R^* is $\mathfrak{p}_{ij}$; w_{ij} induces in $\mathcal{K}(R)$ the discrete normalized valuations v_i of rank 1 such that $\mathcal{C}(v_i/R) = \mathfrak{p}_i$, and v_i stands in this case for the whole set $\{v_{ij}\}$ $(j=1, 2, \cdots)$. Lemmas 3, 5 of [12] imply $l_{ij} = w_{ij}(\zeta_1)$, so that $l_i = v_i(\zeta_1) = w_{ij}(\zeta_1) = l_{ij}$. Hence Lemma 2.1 gives $e(R; \zeta) = e(R^*; \zeta) = \sum_{ij} l_i e(\tau_{ij}R^*; \tau_{ij}\zeta_2, \cdots, \tau_{ij}\zeta_r) = \sum_i l_i e(\tau_i R; \tau_i\zeta_2, \cdots, \tau_i\zeta_r)$, which is the contention.

Case 2 (general case). Let R' be the integral closure of R; if $\mathfrak{m}$ is the maximal prime of R, let $\mathfrak{m}_1', \mathfrak{m}_2', \cdots$ be the minimal primes of $\mathfrak{m}R'$, and set $R_i = R'_{\mathfrak{m}_i'}$, $\mathfrak{m}_i = \mathfrak{m}_i' R_i$. For a given $\mathfrak{p}_i$, let $\mathfrak{p}_{ij}'$ be the minimal primes of $\mathfrak{p}_i R'$, and let $Q(ij)$ be the set consisting of the integers q such that $\mathfrak{p}_{ij}' \subseteq \mathfrak{m}_q'$; set $\mathfrak{p}_{ijq} = \mathfrak{p}_{ij}' R_q$ for any $q \in Q(ij)$. Let v_{ij} be the nontrivial normalized discrete valuation of rank 1 of $\mathcal{K}(R)$ whose center on R' is $\mathfrak{p}_{ij}'$. According to case 1 we have $e(R_q; \zeta) = \sum^{(q)} v_{ij}(\zeta_1) e(\tau_{ijq}R_q; \tau_{ijq}\zeta_2, \cdots, \tau_{ijq}\zeta_r)$, τ_{ijq} having an evident meaning; $\sum^{(q)}$ denotes summation over all the values of i, j for which $q \in Q(ij)$. If k, k_q stand for $R/\mathfrak{m}$, $R_q/\mathfrak{m}_q$ respectively, Lemma 2.2 and the previous formula give

$$\begin{aligned} e(R; \zeta) &= \sum_q [k_q : k] e(R_q; \zeta) \\ &= \sum_{ij} v_{ij}(\zeta_1) \sum_{q \in Q(ij)} [k_q : k] e(\tau_{ijq}R_q; \tau_{ijq}\zeta_2, \cdots, \tau_{ijq}\zeta_r). \end{aligned}$$

We can now replace, in Lemma 2.2, R by $\tau_i R$ and the R_i's by the $\tau_{ijq}R_q$'s (i, j fixed, $q \in Q(ij)$), obtaining

$$\begin{aligned} &\sum_{q \in Q(ij)} [k_q : k] e(\tau_{ijq}R_q; \tau_{ijq}\zeta_2, \cdots, \tau_{ijq}\zeta_r) \\ &\qquad = [K_{v_{ij}} : \mathcal{K}(R/\mathfrak{p}_i)] e(\tau_i R; \tau_i\zeta_2, \cdots, \tau_i\zeta_r). \end{aligned}$$

This and the previous formula give

$$\begin{aligned} e(R; \zeta) &= \sum_{ij} v_{ij}(\zeta_1) [K_{v_{ij}} : \mathcal{K}(R/\mathfrak{p}_i)] e(\tau_i R; \tau_i\zeta_2, \cdots, \tau_i\zeta_r) \\ &= \sum_i l_i e(\tau_i R; \tau_i\zeta_2, \cdots, \tau_i\zeta_r), \qquad \text{Q.E.D.} \end{aligned}$$

THEOREM 2.1 (ASSOCIATIVITY FORMULA). *Let R be a geometric domain, $\{\zeta_1, \cdots, \zeta_r\}$ a set of parameters of R; for $0 \leqq s \leqq r$ denote by $\mathfrak{p}_1, \mathfrak{p}_2, \cdots$ the minimal primes of the ideal $\sum_{i=1}^s \zeta_i R$, and by σ_i the homomorphic mapping of R whose kernel is $\mathfrak{p}_i$. Then $\{\sigma_i\zeta_{s+1}, \cdots, \sigma_i\zeta_r\}$ is a set of parameters of $\sigma_i R$, $\{\zeta_1, \cdots, \zeta_s\}$ is a set of parameters of $R_{\mathfrak{p}_i}$, and we have: $e(R; \zeta_1, \cdots, \zeta_r) = \sum_i e(R_{\mathfrak{p}_i}; \zeta_1, \cdots, \zeta_s) e(\sigma_i R; \sigma_i\zeta_{s+1}, \cdots, \sigma_i\zeta_r)$.*

Proof. The theorem is true for any r and $s=0$. Assume it to be true for all $r < r_0$, and for $r = r_0$ but $s < s_0$; we shall give a proof for $r = r_0$ and $s = s_0 > 0$.

Denote by $\mathfrak{P}_i$ $(i=1, 2, \cdots)$ the minimal primes of $\sum_{j=1}^{s-1}\zeta_j R$, and let $Q(i)$ be the set consisting of the q's such that $\mathfrak{P}_i \subset \mathfrak{p}_q$; let τ_i be the homorphic mapping of R whose kernel is $\mathfrak{P}_i$. For $q \in Q(i)$ let v_{iqj} $(j=1, 2, \cdots)$ be the distinct nontrivial normalized discrete valuations of rank 1 of $\mathfrak{K}(\tau_i R)$ whose center on $\tau_i R$ is $\tau_i \mathfrak{p}_q$. Then, by Lemma 2.3 applied to $\tau_i R$ and by our recurrence assumption, we have

$$\begin{aligned} e(R; \zeta_1, \cdots, \zeta_r) &= \sum_i e(R_{\mathfrak{P}_i}; \zeta_1, \cdots, \zeta_{s-1}) e(\tau_i R; \tau_i\zeta_s, \cdots, \tau_i\zeta_r) \\ &= \sum_i e(R_{\mathfrak{P}_i}; \zeta_1, \cdots, \zeta_{s-1}) \\ &\quad \times \sum_{q \in Q(i)} \{ \sum_j v_{iqj}(\tau_i\zeta_s)[K_{v_{iqj}}:\mathfrak{K}(R/\mathfrak{p}_q)]\} \\ &\quad \times e(\sigma_q R; \sigma_q\zeta_{s+1}, \cdots, \sigma_q\zeta_r) \\ &= \sum_q \{ \sum^{(q)} e(R_{\mathfrak{P}_i}; \zeta_1, \cdots, \zeta_{s-1}) \sum_j v_{iqj}(\tau_i\zeta_s)[K_{v_{iqj}}:\mathfrak{K}(R/\mathfrak{p}_q)]\} \\ &\quad \times e(\sigma_q R; \sigma_q\zeta_{s+1}, \cdots, \sigma_q\zeta_r), \end{aligned}$$

where $\sum^{(q)}$ means summation extended over all the values of i for which $q \in Q(i)$. Now, Lemma 2.3 applied to $\tau_i R_{\mathfrak{p}_q}$ and to its only parameter $\tau_i\zeta_s$ gives $e(\tau_i R_{\mathfrak{p}_q}; \tau_i\zeta_s) = \sum_j v_{iqj}(\tau_i\zeta_s)[K_{v_{iqj}}:\mathfrak{K}(R/\mathfrak{p}_q)]$, so that

$$\begin{aligned} &\sum^{(q)} e(R_{\mathfrak{P}_i}; \zeta_1, \cdots, \zeta_{s-1}) \sum_j v_{iqj}(\tau_i\zeta_s)[K_{v_{iqj}}:\mathfrak{K}(R/\mathfrak{p}_q)] \\ &= \sum^{(q)} e(R_{\mathfrak{P}_i}; \zeta_1, \cdots, \zeta_{s-1}) e(\tau_i R_{\mathfrak{p}_q}; \tau_i\zeta_s) = e(R_{\mathfrak{p}_q}; \zeta_1, \cdots, \zeta_s) \end{aligned}$$

because of our recurrence assumption. Hence $e(R; \zeta_1, \cdots, \zeta_r) = \sum_q e(R_{\mathfrak{p}_q}; \zeta_1, \cdots, \zeta_s) e(\sigma_q R; \sigma_q\zeta_{s+1}, \cdots, \sigma_q\zeta_r)$, Q.E.D.

3. The decomposition theory of a geometric domain.

LEMMA 3.1. *Let R be a geometric domain, $\mathfrak{m}$ its maximal prime. Let x be an indeterminate, and set $R^* = R[x]_{\mathfrak{m}R[x]}$. If $\{\zeta_1, \cdots, \zeta_r\}$ is a set of parameters of R, it is also a set of parameters of R^*, and $e(R; \zeta) = e(R^*; \zeta)$.*

Proof. Assume the result to be true when $0 < \dim R < r$; then, by Theorem 2.1, it is also true when $\dim R = r$. Therefore it is sufficient to prove the result in the case $r=1$. In this case let v_i $(i=1, 2, \cdots)$ be all the distinct nontrivial normalized discrete valuations of rank 1 of $\mathfrak{K}(R) = H$ whose centers on R are $\mathfrak{m}$; according to Lemma 2.3 we have $e(R; \zeta) = \sum_i v_i(\zeta)[K_{v_i}:R/\mathfrak{m}]$. Let v_i^* be the unique extension of v_i to $H(x)$ such that $K_{v_i^*} = K_{v_i}(x')$, where x' is the v_i^*-residue of x and is transcendental over K_{v_i}; then the v_i^*'s are all the distinct nontrivial normalized discrete valuations of rank 1 of $H(x)$ whose centers on R^* are $\mathfrak{m}R^*$, so that $e(R^*; \zeta) = \sum_i v_i^*(\zeta)[K_{v_i^*}:R^*/\mathfrak{m}R^*]$; but $v_i^*(\zeta) = v_i(\zeta)$, $K_{v_i^*} = K_{v_i}(x')$, $R^*/\mathfrak{m}R^* = (R/\mathfrak{m})(x')$, Q.E.D.

Let K be a finite extension of a field H, and let v be a valuation of H;

let N be the smallest normal extension of H containing K; let $\mathfrak{G}$ be the Galois group of N over H, $\mathfrak{H}$ the Galois group of N over K; let $\{\sigma_1, \cdots, \sigma_n\}$ be a set of representatives of the left cosets of $\mathfrak{H}$ in $\mathfrak{G}$. Let w be any extension of v to N, and denote by v_i the valuation of K induced by $\sigma_i^{-1}w$; then each v_i is an extension of v to K, and each extension of v to K is an element of the set $\{v_1, \cdots, v_n\}$ [5]. We contend that this set does not depend on w or on the choice of the σ_i's. In fact, let $\{\tau_i\}$ be another set of representatives of the left cosets of $\mathfrak{H}$ in $\mathfrak{G}$, so that $\tau_i = \sigma_i h_i$ for some $h_i \in \mathfrak{H}$; then $\tau_i^{-1}w = h_i^{-1}\sigma_i^{-1}w$, and this induces v_i in K. On the other hand, if w' is another extension of v to N, we have $w' = \sigma w$ for some $\sigma \in \mathfrak{G}$; then $\sigma_i^{-1}w' = \sigma_i^{-1}\sigma w$, and $\sigma_i^{-1}\sigma \notin \mathfrak{H}\sigma_j^{-1}\sigma$ if $i \neq j$, so that the valuations induced in K by $\{\sigma_i^{-1}\sigma w\}$ are the same as those induced by $\{\sigma_i^{-1}w\}$, possibly in a different order.

The set $\{v_1, \cdots, v_n\}$, which depends only on v, and which is formed by all the extensions of v to K, each repeated a certain number of times, will be called the *complete set of extensions of v to K*. It is easily verified that, for a given extension v' of v to K, the number of v_i's which coincide with v' equals the number of left cosets $\sigma_i\mathfrak{H}$ which contain some element of the decomposition group (on H) of a fixed, but arbitrary, extension of v' to N (see [5]).

The above definition can be extended to the case in which K is any algebraic function field over H: assume $K = H(\xi_1, \cdots, \xi_m)$, and set $d = \text{transc } K/H$; if x_0 is an indeterminate, set $x_i = x_0\xi_i$ $(i = 1, \cdots, m)$, so that $K' = H(x_0, \cdots, x_m)$ is homogeneous in the set $\{x\}$, and K is the set of homogeneous elements of K' of degree zero. Let t_{ij} be indeterminates $(i = 0, \cdots, d;$ $j = 0, \cdots, m)$, and set $y_i = \sum_{j=0}^{m} t_{ij}x_j$, $H^* = H(t)$, $K^* = K'(t)$. Now K^* is a finite extension of $H^*(y)$ by Lemma 2.1 of [1]. If v is a valuation of H, let v^* be the unique extension of v to $H^*(y)$ such that the v^*-residues of $t_{00}, \cdots, t_{dm}, y_0, \cdots, y_d$ are algebraically independent over K_v, and let $\{v_1^*, \cdots, v_n^*\}$ be the complete set of extensions of v^* to K^*; each v_i^* is the unique extension to K^* of a valuation v_i of K, the extension being such that the v_i^*-residues of $t_{00}, \cdots, t_{dm}, y_0$ are algebraically independent over K_{v_i}. The set $\{v_1, \cdots, v_n\}$ is called the *complete set of extensions of v to K with respect to* $\{x_0, \cdots, x_m\}$ or *to* $\{\xi_1, \cdots, \xi_m\}$. It consists of all the extensions of v to K, of dimension $d + \dim v$ (over whatever ground field we choose), each repeated a certain number of times. The definition is justified by the fact that if K is a finite extension of H, then this set coincides with the previously defined complete set of extensions of v to K. We also have that v and each v_i have the same rank.

We now wish to extend these considerations to geometric domains. In the next few lemmas we shall keep the following notations fixed: F is an irreducible variety over the field k, $H = k(F)$, G is an r-dimensional irreducible proper subvariety of F, $R = Q(G/F)$, $\mathfrak{m} = \mathfrak{P}(G/F)$; $K' = H(x_0, \cdots, x_m)$ is an algebraic function field over H, homogeneous in the set $\{x\}$, and K is the field consisting of the homogeneous elements of degree zero of K', so that $H \subseteq K$; d is the transcendency of K over H; t_{ij} $(i = 0, \cdots, d+1; j = 0, \cdots, m)$ are indeterminates, and $y_i = \sum_{j=0}^{m} t_{ij}x_j$; $k^* = k(t)$, $H^* = H(t)$, $K^* = K'(t)$, $R^* = R[t]_{\mathfrak{m}R[t]}$, $\mathfrak{m}^* = \mathfrak{m}R^*$; $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$ are the distinct minimal primes of $\mathfrak{m}^*R^*[x]$; we also set $R_i^* = R^*[x]_{\mathfrak{m}_i}$, $R_i = R_i^* \cap K$; now K^* is a finite extension of $H^*(y_0, \cdots, y_d)$; let N be the smallest normal extension of $H^*(y_0, \cdots, y_d)$ containing K^*, $\mathfrak{G}$ be the Galois group of N over $H^*(y_0, \cdots, y_d)$, $\mathfrak{H}$ the Galois group of N over K^*, $\{\sigma_1, \cdots, \sigma_n\}$ a set of representatives of the left cosets of $\mathfrak{H}$ in $\mathfrak{G}$, $\mathfrak{O} = R^*[\sigma_1 x, \cdots, \sigma_n x]$, so that $N = \mathcal{K}(\mathfrak{O})$. Let $\psi(t, y_0, \cdots, y_d, Y)$ be a polynomial of $H[t, y_0, \cdots, y_d, Y]$, of least degree in the t's, in Y, and in each y_i, such that $\psi(t, y_0, \cdots, y_d, y_{d+1}) = 0$; then ψ is a form in $y_0, \cdots, y_d$, Y and also in the t's, because $\psi = \Psi_{t,y_0,\cdots,y_d,Y}\Delta$, where Δ is the irreducible algebraic correspondence between H and the variety over k whose h.g.p. is $\{x\}$, determined by the embedding of H in K.

Lemma 3.2. *If ψ is such that its coefficients are proportional to elements of R, one of these being a unit, then y_{d+1} and $\mathfrak{O}$ are integrally dependent on $R^*[y_0, \cdots, y_d]$, and conversely.*

Proof. The converse is clear. In order to prove the direct, assume the condition to be fulfilled, and set $t'_{ij} = t_{ij} + \alpha t_{d+1,j}$ $(i = 0, \cdots, d)$, $t'_{d+1,j} = t_{d+1,j}$, where $\alpha \in R$; let y'_i $(i = 0, \cdots, d+1)$ be constructed starting from $\{t'_{ij}\}$ as y_i is from $\{t_{ij}\}$, so that $y'_i = y_i + \alpha y_{d+1}$ $(i = 0, \cdots, d)$, $y'_{d+1} = y_{d+1}$. There exists an isomorphic mapping π of $H^*[y_0, \cdots, y_{d+1}]$ onto itself such that $\pi a = a$ if $a \in H$; $\pi t_{ij} = t'_{ij}$, $\pi y_i = y'_i$. Set $\psi'(t, y_0, \cdots, y_d, Y) = \psi(t', y_0 + \alpha Y, \cdots, y_d + \alpha Y, Y)$. Assume $\psi(t, y_0, \cdots, y_d, Y) = \sum_{\{e\}} a_{e_0, \cdots, e_{d+1}}(t) y_0^{e_0} \cdots y_d^{e_d} Y^{e_{d+1}}$; if g is the degree of ψ in $\{y_0, \cdots, y_d, Y\}$, the coefficient of Y^g in ψ' is $A(t') = \sum_{\{e\}} a_{\{e\}}(t') \alpha^{e_0 + \cdots + e_d}$, and this is a unit of R^* if and only if $A(t)$ is a unit of R^*. Since one of the $a_{\{e\}}(t)$ is a unit of R^* and H is infinite, α can be selected in such a way that $A(t)$ is a unit of R^*. Since $\psi'(t, y_0, \cdots, y_d, y_{d+1}) = \pi\psi(t, y_0, \cdots, y_d, y_{d+1}) = 0$, this proves that y_{d+1} is integrally dependent on $R^*[y_0, \cdots, y_d]$, and that already in $\psi(t, y_0, \cdots, y_d, Y)$ the coefficient of the highest power of Y is a unit of R^*.

Now, consider $\psi(t, y_0, \cdots, y_d, Y)$ as a polynomial in Y; it is possible to find $m+1$ sets $\{t_{d+1,0}^{(i)}, \cdots, d_{d+1,m}^{(i)}\}$ of units of R such that (1) det $(t_{d+1,j}^{(i)})$ is a unit of R, and (2) the coefficient of the highest power of Y in ψ remains a unit of R^* after replacing the $t_{d+1,j}$'s by the $t_{d+1,j}^{(i)}$'s. When such replacement takes place, the relation $\psi(t, y_0, \cdots, y_d, y_{d+1}) = 0$ becomes an equation of integral dependence of $\sum_j t_{d+1,j}^{(i)} x_j$ on $R^*[y_0, \cdots, y_d]$. Therefore there are $m+1$ linear combinations of the x's, with coefficients which are units of R and whose determinant is a unit of R, which are integrally dependent on $R^*[y_0, \cdots, y_d]$. This proves that each x_j is integrally dependent on $R^*[y_0, \cdots, y_d]$, Q.E.D.

Using the same argument we can prove the following result:

Corollary. *If v is any valuation of H over k, and v^* is the unique extension* *of v to H^* over k^*, then each x_i is integrally dependent on $R_{v^*}[y_0, \cdots, y_d]$. As a consequence, each x_i is integrally dependent on $H^*[y_0, \cdots, y_d]$.*

LEMMA 3.3. *Let v be an r-dimensional valuation of H over k whose center on R is $\mathfrak{m}$; let v^* be the unique extension of v to $H^*(y_0, \cdots, y_d)$ over $k^*(y_0, \cdots, y_d)$, and let w be an extension of v^* to N (or to K^*). Then the center of w on $\mathfrak{O}$ (or on $R^*[x]$) has dimension $r+d+1$ over k^*.*

Proof. R_w contains $\mathfrak{O}$ as a consequence of the corollary to Lemma 3.2. We have dim $w/k^*=r+d+1$, hence $\mathcal{C}(w/\mathfrak{O})$ has dimension not greater than $r+d+1$; but since it lies over $\mathfrak{m}^*R^*[y_0, \cdots, y_d]$ it must have dimension not less than $r+d+1$, Q.E.D.

LEMMA 3.4. *For any r-dimensional valuation v of H over k with center $\mathfrak{m}$ on R, let v^* be its unique extension to $H^*(y_0, \cdots, y_d)$ over $k^*(y_0, \cdots, y_d)$; for a given i, denote by $U(i)$ the set of those v's such that v^* has some extension w to K^* for which $\mathfrak{m}_i\subseteq\mathcal{C}(w/R^*[x])$, and let $M(i, v)$ be the set of such w's; then:*

(1) $\mathfrak{m}_i=\bigcap_{v\in U(i)}\bigcap_{w\in M(i,v)}\mathcal{C}(w/R^*[x])$;

(2) *each $\mathfrak{m}_i$ lies on $\mathfrak{m}^*R^*[y_0, \cdots, y_d]$ and has dimension not less than $r+d+1$;*

(3) *for each i, there is some prime of $\mathfrak{O}$ which lies on $\mathfrak{m}_i$.*

Proof. Assume (1) to be true; then $\mathfrak{m}^*R^*[y_0, \cdots, y_d]\subseteq\mathfrak{m}_i\subseteq\mathcal{C}(w/R^*[x])$ for some w; hence dim $\mathfrak{m}_i\geqq$dim $\mathcal{C}(w/R^*[x])=r+d+1$ by Lemma 3.3; also $\mathfrak{m}^*R^*[y_0, \cdots, y_d]\subseteq\mathfrak{m}_i\cap R^*[y_0, \cdots, y_d]\subseteq\mathcal{C}(w/R^*[y_0, \cdots, y_d])$ $=\mathfrak{m}^*R^*[y_0, \cdots, y_d]$. This proves assertion (2).

Again under the assumption that (1) be true, we have, for a given i, $\mathfrak{m}_i\mathfrak{O}\subseteq\bigcap_{v\in U(i)}\bigcap_{w\in M(i,v)}\bigcap_{w'}\mathcal{C}(w'/\mathfrak{O})$, where w' ranges among all the extensions of w to N; hence $\mathfrak{m}_i\mathfrak{O}\cap R^*[x]\subseteq\bigcap_{v\in U(i)}\bigcap_{w\in M(i,v)}\mathcal{C}(w/R^*[x])=\mathfrak{m}_i$; this proves that $\mathfrak{m}_i\mathfrak{O}\cap R^*[x]=\mathfrak{m}_i$, from which (3) follows.

There remains to prove statement (1). This and (2) are equivalent to a part of Theorem 3.5 of [1], and we shall here translate into algebraic terms the proof of Theorem 3.5 of [1].

Set $\mathfrak{m}_i'=\mathfrak{m}_i\cap R[x]$, so that dim $\mathfrak{m}_i'/k\geqq$dim $\mathfrak{m}_i/k^*\geqq r+1$; hence $\mathfrak{m}_i'$ is the intersection of all the $(r+1)$-dimensional primes containing it. If $\mathfrak{P}'$ is any one of these, set $\mathfrak{P}=\mathfrak{P}'R^*[x]$, so that dim $\mathfrak{P}/k^*=r+1$. Let $N(\mathfrak{P})$ be the set of all the r-dimensional valuations v of H having center $\mathfrak{m}$ on R, such that $\mathcal{C}(w/R^*[x])\subseteq\mathfrak{P}$ for some extension w of v^* to K^*. We contend that $N(\mathfrak{P})$ is nonempty: in fact, let w' be an r-dimensional valuation of K whose center on $R[x]$ is $\mathfrak{P}'$; w' induces in H a valuation v of dimension not greater than r, whose center on R is $\mathfrak{m}$; hence dim $v=r$. Let v' be the unique extension of v to H^* over k^*, w^* the unique extension of w' to K^* over k^*; then $R_{v'}[y_0, \cdots, y_d]\subseteq R_{w^*}$, and $\mathfrak{P}_{w^*}$ contains $\mathfrak{P}_{v'}[y_0, \cdots, y_d]$, and therefore it also contains some minimal prime $\mathfrak{p}'$ of $\mathfrak{P}_{v'}[x]$. Let S be the integral closure of $R_{v'}[x]$, $\mathfrak{p}$ one of the minimal primes of $\mathfrak{p}'S$. By the corollary to Lemma 3.2 the x's are integrally dependent on $R_{v'}[y_0, \cdots, y_d]$, so that S is the integral closure in K^* of $R_{v'}[y_0, \cdots, y_d]$; if T is the multiplicatively closed set $R_{v'}[y_0, \cdots, y_d] - \mathfrak{P}_{v'}[y_0, \cdots, y_d]$, we have that S_T is the integral closure of $R_{v'}[y_0, \cdots, y_d]_T = R_{v^*}$, and $\mathfrak{p}S_T$ is a minimal prime of $\mathfrak{P}_{v^*}$; hence $S_{\mathfrak{p}} \subseteq R_w$ for some extension w of v^* to K^*, and $\mathcal{C}(w/R^*[x]) = \mathfrak{p}' \cap R^*[x] \subseteq \mathcal{C}(w^*/R^*[x]) = \mathfrak{P}$. This proves that $v \in N(\mathfrak{P})$.

For $v \in N(\mathfrak{P})$ set $\mathfrak{D}(v)$ = intersection of those $\mathcal{C}(w/R^*[x])$ such that w is related to v as previously stated, and that $\mathcal{C}(w/R^*[x]) + \mathfrak{m}_i \subseteq \mathfrak{P}$; set also $M = \bigcup_{\mathfrak{p} \supseteq \mathfrak{m}_i} N(\mathfrak{P})$, $\mathfrak{A} = \bigcap_{v \in M} \mathfrak{D}(v)$. Since $\mathfrak{A} \subseteq \mathfrak{P}$ for each $\mathfrak{P}$, we have $\mathfrak{A} \subseteq \mathfrak{m}_i$, so that $\mathfrak{m}_i$ contains some minimal prime $\mathfrak{B}$ of $\mathfrak{A}$; clearly there exist a subset M' of M and for each $v \in M'$ an intersection $\mathfrak{D}'(v)$ of components of $\mathfrak{D}(v)$, such that $\mathfrak{B} = \bigcap_{v \in M'} \mathfrak{D}'(v)$. Each $\mathfrak{D}'(v)$ contains $\mathfrak{m}^* R^*[x]$, hence the same is true of $\mathfrak{B}$, which proves that $\mathfrak{B} = \mathfrak{m}_i$; this completes the proof, since $U(i) = M'$ and $\mathfrak{D}'(v) = \bigcap_{w \in M(i,v)} \mathcal{C}(w/R^*[x])$, Q.E.D.

We shall now give the following definition: let, say, R_1 have the same dimension as R; for a given r-dimensional valuation v of H whose center on R is $\mathfrak{m}$, let $n_1(v)$ be the number of elements of the complete set of extensions of v to K with respect to $\{x\}$ whose valuation rings contain R_1; if $n_1(v)$ does not depend on v, it will be denoted by $e(R_1/H; x)$.

THEOREM 3.1. *Let F be an irreducible variety over k, G an irreducible r-dimensional subvariety of F, $R = Q(G/F)$, $H = k(F)$. Let D be an irreducible algebraic correspondence between F and an irreducible variety V over k, $\Delta = D[F]$, $\{x\}$ the h.g.p. of Δ, $K = H(\Delta)$. Let D_1 be a component of $[D; V, G]$ which has the dimension $r + \dim \Delta$ (if there is any); set $R_1 = Q(D_1/D)$, $\Delta_1 = D_1[G]$. If v is a valuation of H whose center on F is G, let $C(v)$ be the complete set of extensions of v to K with respect to $\{x\}$, and let $n(v)$ be the number of elements of $C(v)$ which have the center D_1 on D. Then we have:*

1. *$e(R_1/H; x)$ exists if and only if $n(v)$ does not depend on v when v is a prime divisor of H;*
2. *if $e(R_1/H; x)$ exists, then $e(R_1/H; x) = n(v)$ for any v;*
3. *if $e(R_1/H; x)$ exists, and if $\{\zeta\}$ is a set of parameters of R, then*

$$\frac{e(R_1; \zeta)}{e(R; \zeta)} = \frac{e(R_1/H; x) \text{ ins } \Delta}{\text{ord } \Delta_1};$$

4. *if F is analytically irreducible at G, then $e(R_1/H; x)$ exists;*
5. *let $v_1, \cdots, v_l$ be the distinct elements of $C(v)$ whose center on D is D_1 (l will generally depend on v); then, if v has the dimension r, we have*

$$n(v) \text{ ins } \Delta[K_v : k(G)] = \text{ord } \Delta_1 \sum_{j=1}^{l} [\Gamma_{v_j} : \Gamma_v][K_{v_j} : k(D_1)].$$

Proof. Let $R^*, y_0, \cdots, y_{d+1}, H^*, \cdots$ have the previously stated meanings. D_1 corresponds to an $\mathfrak{m}_i$, say $\mathfrak{m}_1$, which has the dimension $r+d+1$; by Lemma 3.1, $e(R_1; \zeta) = e(R^*[x]_{\mathfrak{m}_1}; \zeta)$, and $e(R; \zeta) = e(R^*; \zeta) = e(\mathfrak{o}; \zeta)$, having set $\mathfrak{o} = R^*[y_0, \cdots, y_d]_{\mathfrak{m}^* R[y_0, \cdots, y_d]}$. If v is a valuation of H, we shall consistently denote by v^* its unique extension to $H^*(y_0, \cdots, y_d)$ over $k^*(y_0, \cdots, y_d)$; if u is any extension of v^* to N, $n(v)$ equals the number of valuations $\sigma_j^{-1}u$ $(j=1, \cdots, n)$ whose center on $R^*[x]$ is $\mathfrak{m}_1$. An extension w of v to K has center D_1 on D if and only if its unique extension w^* to K^* over k^* has the center $\mathfrak{m}_1$ on $R^*[x]$; and this will be the case as soon as $\mathfrak{m}_1 \subseteq \mathcal{C}(w^*/R^*[x])$ (see Lemma 3.4).

The proof will be achieved in several steps, the numbering of the steps having no relation to the numbering of the contentions.

Step 1. Assume $e(R_1/H; x)$ to exist, and let v be a valuation of H whose center on F is G, so that dim $v \geqq r$; let v' be an r-dimensional valuation of H, compounded with v, and whose center on F is G (such v' certainly exists); then there is a 1-1 correspondence $w_i \rightarrow w_i'$ between $C(v)$ and $C(v')$ such that w_i' is compounded with w_i, and we have $\mathcal{C}(w_i/R[x]) \subseteq \mathcal{C}(w_i'/R[x])$. If i is such that $\mathcal{C}(w_i/R[x]) = \mathfrak{m}_1 \cap R[x]$, then $\mathfrak{m}_1 \cap R[x] \subseteq \mathcal{C}(w_i'/R[x])$, hence $\mathfrak{m}_1 \cap R[x] = \mathcal{C}(w_i'/R[x])$. Conversely, if $\mathcal{C}(w_i'/R[x]) = \mathfrak{m}_1 \cap R[x]$, then $\mathcal{C}(w_i/R[x]) \subseteq \mathfrak{m}_1 \cap R[x]$, $\mathcal{C}(w_i^*/R^*[x]) \subseteq \mathfrak{m}_1$; but $\mathfrak{m}^* R^*[x] \subseteq \mathcal{C}(w_i^*/R^*[x])$, or finally $\mathcal{C}(w_i^*/R^*[x]) = \mathfrak{m}_1$. This shows that $n(v) = n(v') = e(R_1/H; x)$, which proves assertion 2 and a part of assertion 1.

Step 2. From the theory of the decomposition group of a valuation as given in [5], [6], one derives that the number n_j of elements of $C(v)$ which coincide with a given $v_j \in C(v)$ fulfills the relation

$$n_j \text{ ins } \Delta = [\Gamma_{v_j^*} : \Gamma_{v^*}][K_{v_j^*} : K_{v^*}]; \tag{2}$$

now, if dim $v^*/k^* = r$ we have $[K_{v_j^*} : K_{v^*}][K_v : k(G)] = [K_{v_j} : k(D_1)]$ ord Δ_1, from which statement 5 follows.

Step 3. Let F be analytically irreducible at G, so that R, R^*, and $\mathfrak{o}$ are analytically irreducible; set $\mathfrak{o}^* = R_1^* = R^*[x]_{\mathfrak{m}_1}$; then $\mathfrak{o} \subseteq \mathfrak{o}^*$ because $\mathfrak{m}_1$ lies on $\mathfrak{m}^* R^*[y_0, \cdots, y_d]$ by Lemma 3.4. Let v be a prime divisor of H whose center on R is $\mathfrak{m}$, and denote by $\bar{\mathfrak{o}}$, $\bar{\mathfrak{o}}^*$ the completions of $\mathfrak{o}$, $\mathfrak{o}^*$ respectively, so that $\bar{\mathfrak{o}}$ is an integral domain. By the same argument used in the proof of Theorem 4 of [13] we obtain that: $\bar{\mathfrak{o}}^*$ is a finite $\bar{\mathfrak{o}}$-module, $\mathfrak{o}^*$ is integrally dependent on $\bar{\mathfrak{o}}$, and v^* has a well determined extension $\bar{v}$ to $\mathcal{K}(\bar{\mathfrak{o}})$ whose center on $\bar{\mathfrak{o}}$ is $\mathfrak{m}^* \bar{\mathfrak{o}}$. Let S be the integral closure of $\mathfrak{o}^*$, $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ the minimal primes of $\mathfrak{m}_1 S$, and set $S_i = S_{\mathfrak{M}_i}$; denote by $\bar{S}_i$ the completion of S_i. Since $\mathfrak{m}^* S_i$ is a primary belonging to the maximal prime of S_i, and such maximal prime has the dimension $r+d+1$ over k^*, we have again that each $\bar{S}_i$ is a finite $\bar{\mathfrak{o}}$-module, so that $\bar{v}$ has finitely many extensions $\bar{v}_{ij}$ to $\mathcal{K}(\bar{S}_i)$, and $\bar{S}_i \subseteq R_{\bar{v}_{ij}}$ for each j. Each $\bar{v}_{ij}$ induces in K^* an extension v_{ij}^* of v^* to K^* such that $S_i \subseteq R_{v_{ij}^*}$ and $\mathcal{C}(v_{ij}^*/S_i) = \mathfrak{M}_i S_i$, or also $\mathcal{C}(v_{ij}^*/R^*[x]) = \mathfrak{m}_1$; clearly, since $\mathcal{C}(v_{ij}^*/S) = \mathfrak{M}_i$, if $i \neq h$ no v_{ij}^* coincides with any v_{hj}^*. Conversely, if w^* is any extension of v^* to K^* whose center on $R^*[x]$ is $\mathfrak{m}_1$, then $\mathcal{C}(w^*/S) = \mathfrak{M}_i$ for some i, and w^* can be extended to a $\bar{w}$ of $\mathcal{K}(\bar{S}_i)$ (since $\bar{S}_i$ is an integral domain); $\bar{w}$ induces in $\mathcal{K}(\bar{\mathfrak{o}})$ the valuation $\bar{v}$, which proves that $\bar{w}$ is one of the $\bar{v}_{ij}$'s, and that w^* is one of the v_{ij}^*'s.

We now have $[\mathcal{K}(\bar{S}_i):\mathcal{K}(\bar{\mathfrak{o}})] = \sum_j [\Gamma_{\bar{v}_{ij}}:\Gamma_{\bar{v}}][K_{\bar{v}_{ij}}:K_{\bar{v}}] = \sum_j [\Gamma_{v_{ij}^*}:\Gamma_{v^*}] \times [K_{v_{ij}^*}:K_{v^*}]$; then $e(\bar{S}_i;\zeta)[S_i/\mathfrak{M}_i S_i:\mathfrak{o}/\mathfrak{m}^*\mathfrak{o}] = [\mathcal{K}(\bar{S}_i):\mathcal{K}(\bar{\mathfrak{o}})]e(\mathfrak{o};\zeta)$, which, together with the previous equality, yields $e(\mathfrak{o};\zeta)\sum_j[\Gamma_{v_{ij}^*}:\Gamma_{v^*}][K_{v_{ij}^*}:K_{v^*}] = e(S_i;\zeta)[S_i/\mathfrak{M}_i S_i:\mathfrak{o}/\mathfrak{m}^*\mathfrak{o}]$; summing with respect to i, and by Lemma 2.2, we obtain: $e(\mathfrak{o};\zeta)\sum_{ij}[\Gamma_{v_{ij}^*}:\Gamma_{v^*}][K_{v_{ij}^*}:K_{v^*}] = [\mathfrak{o}^*/\mathfrak{m}_1\mathfrak{o}^*:\mathfrak{o}/\mathfrak{m}^*\mathfrak{o}]e(\mathfrak{o}^*;\zeta)$.

We now have $[\mathfrak{o}^*/\mathfrak{m}_1\mathfrak{o}^*:\mathfrak{o}/\mathfrak{m}^*\mathfrak{o}] = \text{ord } \Delta_1$, and, by formula (2), $\sum_{ij}[\Gamma_{v_{ij}^*}:\Gamma_{v^*}][K_{v_{ij}^*}:K_{v^*}] = n(v) \text{ ins } \Delta$, so that $n(v) \text{ ins } \Delta e(R;\zeta) = \text{ord } \Delta_1 e(R_1;\zeta)$. This proves statement 3 under the stronger assumption that F be analytically irreducible at G, and also proves that $n(v)$ does not depend on v.

Step 4. Assume $n(v)$ to be independent of v when v is a prime divisor of H; let $\{a_0, \cdots, a_\mu\}$ be the h.g.p. of F, and assume G to be at finite distance for a_0; set $\alpha_i = a_i a_0^{-1}$. Let F_1 be the model of H whose h.g.p. is $\{a_0c_0, \cdots, a_0c_\nu, \cdots, a_\mu c_\nu\}$, the c_i's being forms in the a's proportional to the coefficients of $\Psi_{t,y}\Delta$; let F' be a normal associate to F_1. Let v be an r-dimensional valuation of H over k with center G on F; if $G' = \mathcal{C}(v/F')$, since v is at finite distance for a_0 we have that $Q(G'/F')$ contains $k[\alpha]$ and therefore contains R; hence $r \leqq \dim G' \leqq \dim v = r$, or $\dim G' = r$. Assume v to be at finite distance for, say, c_0; then $c_i c_0^{-1} \in R' = Q(G'/F')$. If R'^*, $\mathfrak{O}'$ are constructed from R' as R^*, $\mathfrak{O}$ have been constructed from R, the last remark, in view of Lemma 3.2, proves that each x and $\mathfrak{O}'$ are integrally dependent(6) on $R'^*[y_0, \cdots, y_d]$. Let v' be a prime divisor of H whose center on F' is G', and construct, as usual, v^* and v'^*; let w be any extension of v^* to N, $\mathfrak{P} = \mathcal{C}(w/\mathfrak{O}')$. Then $\dim \mathfrak{P}/k^* = r+d+1$, and, because of the integral dependence, $\mathfrak{P}$ is a minimal prime of $\mathfrak{m}^*\mathfrak{O}'$, so that (by the same argument used in the proof of step 3) there exists an extension w' of v'^* to N whose center on $\mathfrak{O}'$ is $\mathfrak{P}$; since all the extensions of v^* (of v'^*) are conjugate to w (to w') in $\mathfrak{G}$, we have that the sets of the centers on $\mathfrak{O}'$, hence also on $\mathfrak{O}$, of the extensions of v^*, v'^* to N coincide. This proves that $n(v) = n(v')$; but, by assumption, $n(v')$ does not depend on v', so that $n(v)$ does not depend on v, that is, $e(R_1/H; x)$ exists. Statement 1 is thus completely proved.

Step 5. Assume $e(R_1/H; x)$ to exist. Let S be the integral closure of R, $\mathfrak{p}_1, \cdots, \mathfrak{p}_\mu$ the minimal primes of $\mathfrak{m}S$, and set $S_i = S_{\mathfrak{p}_i}$. From each S_i and from S construct S_i^*, S^* in the same manner as R^* is constructed from R. Given the prime divisor v of H whose center on F is G, the center of v on S will be some $\mathfrak{p}_i$; let $\mathfrak{P}_{i1}, \cdots, \mathfrak{P}_{il_i}$ be those, among the distinct minimal primes of $\mathfrak{p}_i S_i^*[x]$, which lie on $\mathfrak{m}_1$ (and which therefore have dimension r); they are all the centers on $S_i^*[x]$ of those extensions of v to K^* over $k^*(y_0, \cdots, y_d)$ which have center $\mathfrak{m}_1$ on $R^*[x]$. Therefore, if n_{ij} is the number of elements of $C(v)$ which have the center $\mathfrak{P}_{ij}$ on $S_i^*[x]$, we have $\sum_{j=1}^{l_i} n_{ij} = e(R_1/H; x)$. Since now S_i is analytically irreducible, by step 3 we have

(6) This argument is equivalent to the second proof of Theorem 4.3 of [1].

n_{ij} ins $\Delta e(S_i;\zeta) = [S_{ij}/\mathfrak{P}_{ij}S_{ij}:S_i^*/\mathfrak{p}_iS_i^*]e(S_{ij};\zeta)$, after putting $S_{ij}=S_i^*[x]_{\mathfrak{P}_{ij}}$. Summation with respect to j yields

$$e(R_1/H;x)\text{ ins }\Delta e(S_i;\zeta) = \sum_{j=1}^{l_i}[S_{ij}/\mathfrak{P}_{ij}S_{ij}:S_i^*/\mathfrak{p}_iS_i^*]e(S_{ij};\zeta)$$

$$= \frac{\text{ord }\Delta_1}{[S_i/\mathfrak{p}_iS_i:R/\mathfrak{m}]}\sum_j [S_{ij}/\mathfrak{P}_{ij}S_{ij}:K(R^*[x]/\mathfrak{m}_1)]e(S_{ij};\zeta),$$

and summing with respect to i:

$$e(R_1/H;x)\text{ ins }\Delta\sum_{i=1}^{\mu}[S_i/\mathfrak{p}_iS_i:R/\mathfrak{m}]e(S_i;\zeta)$$

$$= \text{ord }\Delta_1\sum_{i=1}^{\mu}\sum_j [S_{ij}/\mathfrak{P}_{ij}S_{ij}:K(R^*[x]/\mathfrak{m}_1)]e(S_{ij};\zeta).$$

Now, the ideals $\mathfrak{P}_{ij}\cap S^*[x]$ $(i=1,\cdots,\mu;j=1,\cdots,l_i)$ are all the distinct minimal primes of $\mathfrak{m}_1S^*[x]$, so that Lemmas 2.2, 3.1 imply $e(R_1/H;\ x)$ $\times$ins $\Delta e(R;\zeta)=\text{ord }\Delta_1 e(R_1;\zeta)$. This completes the proof of statement 3.

Step 6. Let F be analytically irreducible at G. Then, by step 3, $n(v)$ does not depend on v when v is a prime divisor of H; we can then apply step 4, obtaining that $e(R_1/H;\ x)$ exists. This proves statement 4, Q.E.D.

If F is not analytically irreducible at G, then $e(R_1/H;\ x)$ generally does not exist; it may happen, however, that $e(R_1/H;\ x)$ exists for some special D even if F is not analytically irreducible at G. This is the case, for instance, when $\{D;\ V, G\}$ exists (see Theorem 5.2); a general result in this direction is given in Theorem 5.5.

4. **The reduction theorem.** Let D be an algebraic correspondence between the irreducible varieties F, V over k, every component of which operates on the whole F; let G be an irreducible subvariety of F such that $\{D;\ V, G\}$ exists; if D' is a component of $\{D;\ V, G\}$, the coefficient of D' in the expression of $\{D;\ V, G\}$ or of $\{D;\ V, G\}^*$ is called, respectively, the *multiplicity of D' in $\{D;\ V, G\}$* or *in $\{D;\ V, G\}^*$*. A similar definition applies for the multiplicity of a component Δ' of $\Delta\{v\}$ or of $\Delta\{v\}^*$.

THEOREM 4.1. *Let D be an irreducible algebraic correspondence between the irreducible varieties F, V over k, operating on the whole F. Let G be an irreducible subvariety of F such that $D\{G\}$ exists; let $\{\zeta_1,\cdots,\zeta_\nu\}$ be a set of parameters of $Q(G/F)$; let D' be a component of $[D;\ V, G]$, α the multiplicity of D' in $\{D;\ V, G\}^*$. Then $\{\zeta\}$ is a set of parameters of $Q(D'/D)$, and*

$$\alpha = \frac{e(Q(D'/D);\zeta)}{e(Q(G/F);\zeta)}\cdot$$

Proof. Set $H=k(F)$, $R=Q(G/F)$, $\mathfrak{m}=\mathfrak{P}(G/F)$. Let $\{x_0,\cdots,x_m\}$ be the h.g.p. of a component of V_H of which $D\{F\}$ is a subvariety. Let ψ be a determination of $\Psi_{t,y}D\{F\}$ which, considered as a polynomial of $H[t, y]$, has all its coefficients in R, one of them being $=1$; this is possible because $D\{G\}$ exists. Set $R^* = R[t]_{\mathfrak{m}R[t]}$, $\mathfrak{m}^* = \mathfrak{m}R^*$. Denote by τ the homomorphic mapping of $R^*[y]$ whose kernel is $\mathfrak{m}^*R^*[y]$. Then D' corresponds to an irreducible factor ϕ of $\tau\psi$ in the following manner: D' corresponds to ϕ if $\Psi_{t,y}D'[G]$ is a power of ϕ, say ϕ^μ; and in this case $\alpha = h(D[F])(h(D'[G]))^{-1}\mu$. ϕ corresponds, in turn, to a minimal prime $\mathfrak{M}$ of $\mathfrak{m}^*R^*[y]+\psi R^*[y]$; set $S = R^*[y]_{\mathfrak{M}}$; then S has the set of parameters $\{\psi, \zeta_1, \cdots, \zeta_\nu\}$. Set $\mathfrak{p} = \text{rad } \psi S$, $\mathfrak{P} = \text{rad } \mathfrak{m}^*S$; then $\mathfrak{p}$ and $\mathfrak{P}$ are prime ideals; the homomorphic mapping whose kernel is $\mathfrak{P}$ is an extension of τ, and will be denoted by τ; denote by σ the homomorphic mapping whose kernel is $\mathfrak{p}$. Theorem 2.1 gives:

$$e(S; \zeta, \psi) = e(S_{\mathfrak{P}}; \zeta)e(\tau S; \tau\psi).$$

Now, $S_{\mathfrak{P}} = R^*[y]_{\mathfrak{m}^*R^*[y]}$ and, by Lemma 3.1, $e(S_{\mathfrak{P}}; \zeta) = e(R; \zeta)$. On the other hand, $(\tau\psi)\phi^{-\mu}$ is a unit of τS, so that $e(\tau S; \tau\psi) = e(\tau S; \phi^\mu)$. Since τS is the quotient ring of the principal ideal $\phi(\tau R^*)[\tau y]$, it is a valuation ring R_v; if v is normalized, then $v(\phi) = 1$. Lemma 2.3 implies then $e(\tau S; \phi^\mu) = \mu$. We have thus proved the formula

$$e(S; \zeta, \psi) = \mu e(R; \zeta).$$

On the other hand, Theorem 2.1 also gives $e(S; \psi, \zeta) = e(S_{\mathfrak{p}}; \psi)e(\sigma S; \sigma\zeta)$ $= e(\sigma S; \sigma\zeta)$ since $e(S_{\mathfrak{p}}; \psi) = 1$ because ψ is a regular parameter of $S_{\mathfrak{p}}$. As a consequence, $e(\sigma S; \sigma\zeta) = \mu e(R; \zeta)$. Since σ induces an isomorphism in R^*, we shall write ζ, R^*, $\mathfrak{m}^*$ in place of $\sigma\zeta$, σR^*, $\sigma\mathfrak{m}^*$. We have that $\sigma\mathfrak{M}$ is a minimal prime of $\mathfrak{m}^*R^*[\sigma y]$ (σ having been extended to the homomorphic mapping of $R^*[x]$ whose kernel is $\wp(D[F]/H[x])H(t)[x] \cap R^*[x])$, and that $(\sigma\mathfrak{M})R^*[\sigma x]$ has exactly one minimal prime $\mathfrak{M}'$; if $S' = R^*[\sigma x]_{\mathfrak{M}'}$, then $S' \cap H((\sigma x_1)(\sigma x_0)^{-1}, \cdots, (\sigma x_m)(\sigma x_0)^{-1}) = Q(D'/D)$, and $e(S'; \zeta) = e(Q(D'/D); \zeta)$. Moreover, by Lemma 3.2 we have that each σx_i is integrally dependent on $R^*[\sigma y]$, so that we can apply Lemma 2.2, obtaining

$$[S'/\mathfrak{M}'S' : \sigma S/\sigma\mathfrak{M}]e(S'; \zeta) = [H(t, \sigma x) : H(t, \sigma y)]e(\sigma S; \zeta).$$

Finally we have: $[S'/\mathfrak{M}'S' : \sigma S/\sigma\mathfrak{M}] = h(D'[G])$, $[H(t, \sigma x) : H(t, \sigma y)] = h(D[F])$. Hence

$$e(Q(D'/D); \zeta)h(D'[G]) = \mu h(D[F])e(R; \zeta), \qquad \text{Q.E.D.}$$

Theorem 4.1 is the generalization of Lemma 5.4 of [1], and also brings to light a flaw in the proof of that lemma; in fact, formula (11) of [1] should read "$e = [\Gamma_{w^*} : \Gamma_{v_0^*}]h$, where $h = [H^*(x) : H^*(y)]$," and this is of immediate proof since $h = [\Gamma_{w'} : \Gamma_{w^*}]$(7). (The false step in deriving the original formula (11) was the unproven assumption $H^* \subseteq k^*(y)$.) As a consequence, in the nota-

(7) The tilde of the original paper is here replaced by a prime (′) for typographical reasons.

tion used in the proof of the lemma, we have $h\Lambda'\{v\}=\Lambda\{v\}$, and not $\Lambda'\{v\}=\Lambda\{v\}$. This does not affect the fact that $\Lambda\{v\}$ describes a pencil. The modifications, on the safe side, needed in the statement of Lemma 5.4 of [1] are the following ones: "$\cdots$ *$k(G(\mathfrak{P}))=H$ if $\mathfrak{P}$ is simple; otherwise H is purely inseparable over $k(G(\mathfrak{P}))$. Conversely, any simple pencil on V can be obtained in the described way.*"

It can now be asked if also Lemma 5.3 of [1] has a generalization. That this is the case will be shown in section 5.

THEOREM 4.2 (REDUCTION THEOREM, ELEMENTARY CASE). *Let V_1, V_2 be irreducible varieties over k, of dimensions r_1, r_2 respectively; let W_i $(i=1, 2)$ be an irreducible s_i-dimensional subvariety of V_i, and let D be an irreducible algebraic correspondence between V_1 and V_2, operating on the whole V_i; let U be a common component of $[[D; V_i, W_j]; W_i, W_j]$ $(i, j=1, 2; i\neq j)$, of dimension* $\dim D+s_1+s_2-r_1-r_2$. *Then U operates on the whole W_1 and W_2. Let $D_1^{(i)}$, $D_2^{(i)}, \cdots$ be the components of $[D; V_j, W_i]$ containing U; then each $D_h^{(i)}$ has the dimension* $\dim D+s_i-r_i$ *and operates on the whole V_j. Assume $\{D; V_j, W_i\}^*$ and $\{D_h^{(i)}; W_j, W_i\}^*$ to exist, and let α_{ih}, β_{ih} be the multiplicities of $D_h^{(i)}$, U in $\{D; V_j, W_i\}^*$, $\{D_h^{(i)}; W_j, W_i\}^*$ respectively. Let $\{\zeta^{(i)}\}$ be any set of parameters of $Q(W_i/V_i)$. Then*

$$\sum_h \alpha_{1h}\beta_{1h} = \sum_h \alpha_{2h}\beta_{2h}$$
$$= e(Q(U/D); \zeta^{(1)}, \zeta^{(2)})e(Q(W_1/V_1); \zeta^{(1)})^{-1}e(Q(W_2/V_2); \zeta^{(2)})^{-1},$$

and this number is called the multiplicity of U in $\{D; W_1, W_2\}^*$.

Proof. The only assertion which needs to be proved is the last one.

Write $D_1, \cdots, D_\nu$ in place of $D_1^{(2)}, D_2^{(2)}, \cdots$; according to Theorem 4.1 we have that α_{2i} is given by $e(Q(W_2/V_2); \zeta^{(2)})^{-1}e(Q(D_i/D); \zeta^{(2)})$. For the same reason, β_{2i} equals $e(Q(W_1/V_1); \zeta^{(1)})^{-1}e(Q(U/D_i); \zeta^{(1)})$, so that $\sum_i\alpha_{2i}\beta_{2i}=e(Q(W_1/V_1); \zeta^{(1)})^{-1}e(Q(W_2/V_2); \zeta^{(2)})^{-1}\sum_{i=1}^{\nu} e(Q(D_i/D); \zeta^{(2)})e(Q(U/D_i); \zeta^{(1)})$. But Theorem 2.1 implies $\sum_{i=1}^{\nu} e(Q(D_i/D); \zeta^{(2)})e(Q(U/D_i); \zeta^{(1)})=e(Q(U/D); \zeta^{(1)}, \zeta^{(2)})$, Q.E.D.

5. The behavior of fundamental points.

THEOREM 5.1. *Let Δ be an irreducible algebraic correspondence between the algebraic function field H over k and the irreducible variety V over k, and let $\{x\}$ be the h.g.p. of Δ; if v is a valuation of H over k, let Δ' be a component of $\Delta[v]$, and let β be the multiplicity of Δ' in $\Delta\{v\}^*$. Let C be the complete set of extensions of v to $H(\Delta)$ with respect to $\{x\}$, and let m be the number of elements of C which are related to Δ'; then*

$$\beta = \frac{m \operatorname{ins} \Delta}{\operatorname{ord} \Delta'}\cdot$$

Proof. Let $\psi(y_0, \cdots, y_d, Y)$ be a determination of $\Psi_{t,y_0,\cdots,y_d,Y}\Delta$ having all the coefficients in R_v, one of them being equal to 1. Set $H^*=H(t)$, $K^*=H^*(x)$, $k^*=k(t)$; let v^*, w be the unique extensions of v to H^*, $H^*(y_0, \cdots, y_d)$ over k^*, $k^*(y_0, \cdots, y_d)$ respectively. Let N be the smallest normal extension of $H^*(y_0, \cdots, y_d)$ containing K^*, $\mathfrak{G}$ the Galois group of N over $H^*(y_0, \cdots, y_d)$, $\mathfrak{H}$ the Galois group of N over K^*; let $\{\sigma_1, \cdots, \sigma_n\}$ be a set of representatives of the left cosets of $\mathfrak{H}$ in $\mathfrak{G}$, σ_1 being in $\mathfrak{H}$. The polynomial $\psi(y_0, \cdots, y_d, Y)$ is irreducible over $H^*(y_0, \cdots, y_d)$, but it acquires the root $Y=y_{d+1}$ in K^*. Hence ψ splits completely in N, namely $\psi(y_0, \cdots, y_d, Y)=a\prod_{i=1}^{n}(Y-\sigma_i y_{d+1})^e$, where $e=\exp\Delta$ and $v^*(a)=0$ (Lemma 3.2). An extension w' of w to K^* is related to Δ' if the following statement is true: let π', π be the homomorphic mappings of $R_{v^*}[x]$, R_{v^*} with kernels $\mathfrak{C}(w'/R_{v^*}[x])$, $\mathfrak{P}_{v^*}$ respectively; let $\phi(y_0, \cdots, y_d, Y)$ be the irreducible factor of $\pi\psi(y_0, \cdots, y_d, Y)$ such that $\Psi_{t,y_0,\cdots,y_d,Y}\Delta'$ is a power of ϕ, say ϕ^α; then $\phi=0$ is, but for a factor in K_{v^*}, the minimal equation of $\pi' y_{d+1}$ over $K_{v^*}(y_0, \cdots, y_d)$ (having identified y_i with πy_i and $\pi' y_i$ for $i=0, \cdots, d$). An equivalent way of looking at the connection between w' and Δ' is the following one: Δ' corresponds to a minimal prime $\mathfrak{P}$ of $\mathfrak{P}_{v^*}R_{v^*}[x]$, the correspondence being such that $\phi=0$ is, but for a factor in K_{v^*}, the minimal equation of y_{d+1} over $H^*(y_0, \cdots, y_d)$ mod $\mathfrak{P}$; then w' is related to Δ' if $\mathfrak{C}(w'/R_{v^*}[x])=\mathfrak{P}$. We also remark that any other minimal prime of $\mathfrak{P}_{v^*}[x]$ induces a distinct ideal in $R_{v^*}[y_0, \cdots, y_{d+1}]$, and that $\beta h(\Delta')=\alpha h(\Delta)$.

This being established, let $w_1, \cdots, w_\nu$ be the distinct extensions of w to K^* related to Δ', that is, having the center $\mathfrak{P}$ on $R_{v^*}[x]$; let u_1 be any extension of w_1 to N, and suppose that the values of j for which $\sigma_j^{-1}u_1$ induces w_i in K^* $(i=1, \cdots, \nu)$ are $\mu_{i-1}+1, \cdots, \mu_i$; we have, in particular, $\mu_0=0$, $\mu_\nu=m$. Let $\mathfrak{Z}$ be the decomposition group of u_1 over $H^*(y_0, \cdots, y_d)$ (see [5] and [6]). For $j=1, \cdots, \mu_1$ we have $\sigma_j^{-1}u_1=hu_1$ for some $h\in\mathfrak{H}$, depending on j; we may assume that $\sigma_1, \cdots, \sigma_{\mu_1}$ have been selected in such a way that the h's are all equal to 1. Then $\sigma_j\in\mathfrak{Z}$ for $j\leqq\mu_1$, and any $z\in\mathfrak{Z}$ can be written in the form $\sigma_j h$ with $j\leqq\mu_1$ and $h\in\mathfrak{H}\cap\mathfrak{Z}$; in particular, we can take $\sigma_1=1$. Set $\overline{R}=R_{v^*}[x]/\mathfrak{P}$, and extend π to a well determined homomorphic mapping π' of $R_{v^*}[x]$ onto $\overline{R}$, so that $\overline{R}=K_{v^*}[\pi' x]$; let now π_1 denote a well determined homomorphic mapping of $\mathfrak{O}=R_{v^*}[\sigma_1 x, \cdots, \sigma_n x]$ onto $\mathfrak{O}/\mathfrak{C}(u_1/\mathfrak{O})=\overline{\mathfrak{O}}$; we shall assume π_1 to be an extension of π', so that $\overline{R}\subseteq\overline{\mathfrak{O}}$. Set $\bar\sigma_j=\pi_1\sigma_j\pi_1^{-1}$ for $j=1, \cdots, \mu_1$, so that $\bar\sigma_j$ is an automorphism of $\overline{\mathfrak{O}}$ over $K_{v^*}[y_0, \cdots, y_d]$.

Now we have to take care of the values of j greater than μ_1. For an $i>1$ (but $\leqq\nu$), and for a $j>\mu_{i-1}$ but $\leqq\mu_i$, we have that $\sigma_j^{-1}u_1=h\sigma_{\mu_{i-1}+1}^{-1}u_1$ for some $h\in\mathfrak{H}$ depending on j, and again the elements $\sigma_{\mu_{i-1}+2}, \cdots, \sigma_{\mu_i}$ can be selected in such a way that $h=1$; we then set $\pi_i=\pi_1\sigma_{\mu_{i-1}+1}$, $\bar\sigma_j=\pi_1\sigma_j\pi_i^{-1}$. Now π_i is another well determined homomorphic mapping of $\mathfrak{O}$ onto $\overline{\mathfrak{O}}$, and its kernel is $\mathfrak{C}(u_i/\mathfrak{O})$, where $u_i=\sigma_{\mu_{i-1}+1}^{-1}u_1$, while $\bar\sigma_j$ is an automorphism of $\overline{\mathfrak{O}}$ over $K_{v^*}[y_0, \cdots, y_d]$; in particular, $\bar\sigma_{\mu_{i-1}+1}=1$.

In the following consideration we let i take any of the values $1, \cdots, \nu$, while j is such that $\mu_{i-1} < j \leqq \mu_i$. If $\sigma \in \mathfrak{Z}$ we have $\sigma\sigma_j u_i = \sigma u_1 = u_1$, hence $\sigma\sigma_j = \sigma_l h$ for some l such that $\mu_{i-1} < l \leqq \mu_i$, and for $h \in \mathfrak{H}$; besides, the mapping $j \to l$ is a 1-1 correspondence; this proves that the elementary symmetric functions constructed with the elements $\sigma_{\mu_{i-1}+1} y_{d+1}^e, \cdots, \sigma_{\mu_i} y_{d+1}^e$ belong to the largest separable extension Z of $H^*(y_0, \cdots, y_d)$ contained in the decomposition field of u_1 over $H^*(y_0, \cdots, y_d)$; in particular, $\prod_{j=\mu_{i-1}+1}^{\mu_i} (Y - \sigma_j y_{d+1})^e \in (\mathfrak{O} \cap Z)[Y]$. It is known (see [5], [6]) that $\pi_1(\mathfrak{O} \cap Z) \subseteq K_w$; therefore

$$\prod_{j=\mu_{i-1}+1}^{\mu_i} (Y - \pi_1 \sigma_j y_{d+1})^e = \prod_{j=\mu_{i-1}+1}^{\mu_i} (Y - \bar{\sigma}_j \pi_i y_{d+1})^e = f_i'(Y)$$

belongs to $K_w[Y]$, or also to $K_{v^*}[y_0, \cdots, y_d, Y]$ since $\bar{\mathfrak{O}}$ is integrally dependent on $K_{v^*}[y_0, \cdots, y_d]$ by the corollary to Lemma 3.2. We shall accordingly denote $f_i'(Y)$ by $f_i(y_0, \cdots, y_d, Y)$. Now, $f_i(y_0, \cdots, y_d, \pi_i y_{d+1}) = 0$ because $\bar{\sigma}_{\mu_{i-1}+1} = 1$; also, $f_i(y_0, \cdots, y_d, \pi' y_{d+1}) = 0$, which in turn implies that f_i is divisible by $\phi(y_0, \cdots, y_d, Y)$; and since f_i is a product of linear factors conjugate to each other over $K_{v^*}(y_0, \cdots, y_d)$, it must have the form $b_i \phi^{\rho_i}$, where $b_i \in K_{v^*}$. We now have:

$$\pi\psi(y_0, \cdots, y_d, Y) = (\pi a) b_1 \cdots b_\nu \phi^{\rho_1 + \cdots + \rho_\nu} \omega(y_0, \cdots, y_d, Y),$$

where $\omega = \prod_{l=m+1}^{n} (Y - \pi_1 \sigma_l y_{d+1})^e$. Assume $\omega(y_0, \cdots, y_d, \pi_1 y_{d+1}) = 0$; then $\pi_1(y_{d+1} - \sigma_l y_{d+1}) = 0$ for some $l > m$, hence $(1 - \sigma_l) y_{d+1} \in \mathfrak{P}_{u_1}$, from which we derive $(1 - \sigma_l) z \in \mathfrak{P}_{u_1}$ for any $z \in R_{v^*}[y_0, \cdots, y_{d+1}]$; in particular, if $z \in \mathcal{C}(\sigma_l^{-1} u_1 / R_{v^*}[y_0, \cdots, y_{d+1}])$ it follows that $\sigma_l z \in \mathfrak{P}_{u_1}$ and $z \in \mathfrak{P}_{u_1}$, which means that $\sigma_l^{-1} u_1$ and u_1 have the same center on $R_{v^*}[y_0, \cdots, y_{d+1}]$, a contradiction because $l > m$. We conclude that ω is not divisible by ϕ, so that $\rho_1 + \cdots + \rho_\nu = \alpha$; on the other hand, $\phi^{\rho_1 + \cdots + \rho_\nu}$ has, by construction, the degree em in Y, which proves that $\alpha \deg \Delta' = em$. This finally gives

$$\beta = \frac{emh(\Delta)}{h(\Delta') \deg \Delta'} = \frac{m \operatorname{ins} \Delta}{\operatorname{ord} \Delta'}, \qquad \text{Q.E.D.}$$

COROLLARY. *Maintain the same notation as in Theorem* 5.1, *and let* $v_1, \cdots, v_\nu$ *be the distinct elements of* C *which are related to* Δ'; *then* $\beta = \sum_{j=1}^{\nu} [\Gamma_{v_j} : \Gamma_v] \times [K_{v_j} : K_v(\Delta')]$.

Proof. By formula (2), and with the notation of the proof of Theorem 5.1, we have $m \operatorname{ins} \Delta = \sum_{j=1}^{\nu} [\Gamma_{w_j} : \Gamma_w][K_{w_j} : K_w] = \sum_j [\Gamma_{w_j} : \Gamma_w][K_{w_j} : K_v(\Delta')(t)] \times \operatorname{ord} \Delta'$, so that $\beta = \sum_j [\Gamma_{w_j} : \Gamma_w][K_{w_j} : K_v(\Delta')(t)]$. This is the contention of the corollary, since $w_1, \cdots, w_\nu$ are the extensions of $v_1, \cdots, v_\nu$ to K^* over $k^*(y_0, \cdots, y_d)$, Q.E.D.

THEOREM 5.2. *Maintain the notation of Theorem* 4.1, *and set* $r = \dim G$; *let* $\{x\}$ *be the h.g.p. of* $D[F]$; *then* $n = e(Q(D'/D)/k(F); x)$ *exists. Let* v *be an* r*-dimensional valuation of* $k(F)$ *with center* G *on* F, *and let* $v_1, \cdots, v_\nu$ *be the distinct valuations of* $k(D)$ *of dimension* $r + \dim D - \dim F$ *which induce* v *in*

$k(F)$ and which have the center D' on D. Then

$$\alpha = \frac{n \text{ ins } D[F]}{\text{ord } D'[G]} = \frac{\sum_{j=1}^{\nu} [\Gamma_{v_j}:\Gamma_v][K_{v_j}: k(D')]}{[K_v: k(G)]}.$$

Proof. (1). If F is analytically irreducible at G, the first contention is a consequence of Theorems 4.1 and 3.1.

(2). In the general case, set $\Delta = D[F]$, $K = K_v$; we have, by definition, that $\Delta\{v\}$ is the extension $\mathfrak{z}_K$ to K of the cycle $\mathfrak{z} = D\{G\}$; therefore, if $\mathfrak{z}'$ is the component of $\mathfrak{z}$ given by $D'\{G\}$, and if α' is the coefficient of $\mathfrak{z}'$ in $\mathfrak{z}$, that is, the multiplicity of D' in $\{D; V, G\}$, $\mathfrak{z}'_K$ has the form $\sum_{j=1}^{\mu} e^{-1}(\exp \mathfrak{z}')\Delta_j$, where Δ_j are components of $\Delta\{v\}$, and $e = \exp \Delta_j$ (the same for each j); Δ_j appears in $\Delta\{v\}$ with the multiplicity $\beta_j' = \alpha' e^{-1} \exp \mathfrak{z}'$. If now α, β_j are the multiplicities of $\mathfrak{z}'$, Δ_j in $D\{G\}^*$, $\Delta\{v\}^*$ respectively, the relation between them is β_j ins $\Delta_j = \alpha$ ins $\mathfrak{z}'$; from this relation and Theorem 5.1 we obtain α ins $\mathfrak{z}'$ red $\Delta_j = m_j$ ins Δ, where m_j is the number of elements of the complete set C of extensions of v to $k(D)$ with respect to $\{x\}$ which are related to Δ_j. Therefore α ins $\mathfrak{z}'$ $\sum_j$ red $\Delta_j =$ ins $\Delta \sum_j m_j$, or α ord $\mathfrak{z}' = n$ ins Δ, where n is the number of elements of C whose valuation ring contains $Q(D'/D)$. This shows that n does not depend on v, that is, that $n = e(Q(D'/D)/k(F); x)$, and also gives the first expression for α. The second expression is a consequence of statement 5 of Theorem 3.1, or also of the corollary to Theorem 5.1, Q.E.D.

Now let F, V, D, G, $\{x\}$ have the same meaning as in Theorem 5.2 or 4.1, but let us not necessarily assume the existence of $\{D; V, G\}$. Let D' be a component of $[D; V, G]$ of dimension equal to dim $D-$dim $F+$dim G, if there is any. If $e(Q(D'/D)/k(F); x)$ exists, then a certain multiplicity can be attached to D' with respect to G and D, namely the number $\alpha^* = e(Q(D'/D)/k(F); x)$ ins $D[F]$ (ord $D'[G])^{-1}$ which, by Theorem 3.1, coincides with $e(Q(D'/D); \zeta)e(Q(G/F); \zeta)^{-1}$ ($\{\zeta\}$ being a set of parameters of $Q(G/F)$) which would give the multiplicity of D' in $\{D; V, G\}^*$ if the latter existed. This number will be called the *multiplicity of D' in $\{D; V, G\}^*$* even if $\{D; V, G\}^*$ does not exist; the number $\alpha = \alpha^* h(D'[G])h(D[F])^{-1}$ will in turn be called the *multiplicity of D' in $\{D; V, G\}$*. The same definition applies to the multiplicities of $D'[G]$ in $D\{G\}^*$ or $D\{G\}$. In order to avoid a notation which needs the use of the quotient rings, we shall put $\alpha^* = e(D'/D; V, G)^* = e(D'/D; G, V)^*$, and $\alpha = e(D'/D; V, G) = e(D'/D; G, V)$. Therefore if $\{D; V, G\}^*$ exists it is given by $\sum_i e(D_i/D; V, G)^* D_i$, where D_i ranges among all the distinct components of $[D; V, G]$. According to Theorem 3.1 we have:

THEOREM 5.3. *Let D be an irreducible algebraic correspondence between the irreducible varieties F, V over k, operating on the whole F; let G be an irreducible subvariety of F, D^* a component of $[D; V, G]$ of dimension equal to* dim D

$-\dim F+\dim G$. *Denote by F' any model of $k(F)$, by G' any irreducible pseudo-subvariety of F' which corresponds to G in the birational correspondence between F and F', by D' the irreducible algebraic correspondence between F' and V, birationally equivalent to D, such that $D'[F']=D[F]$; then*:

1. *If F is analytically irreducible at G, $e(D^*/D;\ V, G)$ exists.*

2. *Assume F' to be such that $D'\{G'\}$ exists for each G' which has the same dimension as G, and that $Q(G/F)\subseteq Q(G'/F')$ for each such G'; write $D'\{G'\}=\mathfrak{z}+\mathfrak{z}'$, $\mathfrak{z}$ being such that its components are components of the extension of $D^*[G]$ over $k(G')$, while $\mathfrak{z}'$ has no component in common with such extension. Then $e(D^*/D;\ V, G)$ exists if and only if* $\deg \mathfrak{z}$ *does not depend on G'; and in this case we have*

$$e(D^*/D;\ V, G) = \deg \mathfrak{z}(\deg D^*[G])^{-1}.$$

The reduction theorem can be extended, with the same proof, in the following way:

THEOREM 5.4 (REDUCTION THEOREM, GENERAL CASE). *In the statement of Theorem 4.2, let us replace the assumption of the existence of $\{D;\ V_j,\ W_i\}^*$ and $\{D_h^{(i)};\ W_j,\ W_i\}^*$ by the assumption that $e(D_h^{(i)}/D;\ V_j,\ W_i)^*$ and $e(U/D_h^{(i)};\ W_j,\ W_i)^*$ exist for each i, h; then $\sum_h e(D_h^{(i)}/D;\ V_j,\ W_i)^* e(U/D_h^{(i)};\ W_j,\ W_i)^*$ does not depend on i, and will be denoted by $e(U/D;\ W_1,\ W_2)^*=e(U/D;\ W_2,\ W_1)^*$. Its value is given by the same expression as in Theorem 4.2.*

As we have previously remarked, there is the possibility that $e(D^*/D;\ V,\ G)$ or $e(D^*/D;\ V,\ G)^*$ (notation as in Theorem 5.3) are not integers; that this occurrence may actually take place is proved by the following example: in the notation of Theorem 5.3, let k be the field obtained by adjunction of an indeterminate a to a given field; let F be the irreducible variety over k whose n.h.g.p. is $\{z_1,\ z_2,\ z_3\}$, where $z_1z_2^2=z_3^2$, and let G be the point of F given by $z_1=a$, $z_2=z_3=0$. Then $k(G)=k$. Let V be the straight line over k whose n.h.g.p. is x, and let Δ be the irreducible algebraic correspondence between $k(F)$ and V such that $\wp(\Delta/k(F)[x])$ is the principal ideal generated by $x+z_3z_2^{-1}$. If $D=D_{\Delta,F}$, $D\{G\}$ does not exist; however, $D[G]$ is the irreducible variety $x^2=a$ over $k=k(G)$, and it has to be counted with the multiplicity $1/2$. In fact, choose for F' the plane whose n.h.g.p. is $\{z_1',\ z_2'\}$, where $z_2'=z_2$, $z_1'z_2=z_3$, so that $z_1'^2=z_1$. Then D' has the n.h.g.p. $\{x,\ z_1',\ z_2'\}$, where $x+z_1'=0$, and G' is given by $z_1'=a^{1/2}$, $z_2'=0$, so that $k(G')=k(a^{1/2})$. Therefore $D'\{G'\}$ exists and equals the integral cycle $\mathfrak{z}=1\Delta'^*$, where Δ'^* is the irreducible variety over $k(a^{1/2})$ given by $x=a^{1/2}$.

There are evident cases in which one can assert a priori that $e(D^*/D;\ V, G)$ or $e(D^*/D;\ V,\ G)^*$ are integers. For instance: if $\{D;\ V,\ G\}$ exists, then $e(D^*/D;\ V,\ G)$ is an integer; if G is a point and k is algebraically closed, then $e(D^*/D,\ V,\ G)$ and $e(D^*/D;\ V,\ G)^*$ are integers, as a consequence of statement 2 of Theorem 5.3; if $e(D^*/D;\ V,\ G)$ is an integer and $h(D^*[G])=1$, then $e(D^*/D;\ V, G)^*$ is an integer; if G is simple on F, then $e(D^*/D;\ V, G)^*$ is an integer. Less trivial cases are described in the following result:

THEOREM 5.5. *Maintain the notation of Theorem* 5.3; *we have*:

1. *Let* $\mathfrak{r}$ *be any minimal prime of the zero ideal of the completion* R *of* $Q(G/F)$, *and let* $\{\zeta\}$ *be a set of parameters of* $Q(G/F)$; *then* $e(D^*/D;\ V, G)^*$ *exists if and only if* R *is* (*but for an isomorphism*) *a subring and subspace of the completion* S *of* $Q(D^*/D)$, *and if in addition* $e(S/\mathfrak{r}S;\zeta)e(R/\mathfrak{r};\zeta)^{-1}$ *does not depend on* $\mathfrak{r}$;

2. *If* $e(D^*/D;\ V, G)^*$ *exists, it is not greater than the length* l *of* $\mathfrak{P}(G/F)Q(D^*/D)$; *in particular, if it is known that* $e(D^*/D;\ V, G)^*$ *has to be an integer, and if* $l=1$, *then* $e(D^*/D;\ V, G)^*=1$;

3. *If* G, D^* *are simple on* F, D *respectively, then* $e(D^*/D;\ V, G)^*$ *exists and equals the length of* $\mathfrak{P}(G/F)Q(D^*/D)$;

4. *If* $e(D^*/D;\ V, G)^*$ *exists, it equals the limit, for* j *approaching infinity, of the ratio*

$$\frac{\text{length } (\mathfrak{P}(G/F)^jQ(D^*/D))}{\text{length } \mathfrak{P}(G/F)^j}.$$

Proof. 1. We identify D, D^* respectively with D, D_1 of Theorem 3.1, and let $\mathfrak{o}$, $\mathfrak{o}^*$, $\bar{\mathfrak{o}}$, $\bar{\mathfrak{o}}^*$ have the same meaning as in step 3 of the proof of that theorem. Let $\mathfrak{m}$, $\mathfrak{m}^*$ be the maximal primes of $\mathfrak{o}$, $\mathfrak{o}^*$ respectively, $\bar{\mathfrak{m}}=\mathfrak{m}\bar{\mathfrak{o}}$, $\bar{\mathfrak{m}}^*=\mathfrak{m}^*\bar{\mathfrak{o}}^*$.

Assume $e(D^*/D;\ V, G)^*$ to exist, and let $\mathfrak{O}$ be the integral closure of $\mathfrak{o}$, $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\cdots$ the minimal primes of $\mathfrak{m}\mathfrak{O}$, T_j the $\mathfrak{O}_{\mathfrak{m}_j}$-topology; set $\mathfrak{O}^*=\mathfrak{O}\mathfrak{o}^*$, and let $\mathfrak{m}_1^*$, $\mathfrak{m}_2^*$, $\cdots$ be the minimal primes of $\mathfrak{m}^*\mathfrak{O}^*$, T_j^* the $\mathfrak{O}^*_{\mathfrak{m}_j^*}$-topology. In the course of the proof of Theorem 3.1 it has been shown that for each j there is an $\mathfrak{m}_h^*$ which lies over $\mathfrak{m}_j$, so that T_h^* induces T_j in $\mathfrak{O}_{\mathfrak{m}_j}$ by [13]. Let T, T^* be the $\mathfrak{o}$-topology and the $\mathfrak{o}^*$-topology respectively; then T^* is induced in $\mathfrak{o}^*$ by $T_1^*\cap T_2^*\cap\cdots$, and T is induced in $\mathfrak{o}$ by $T_1\cap T_2\cap\cdots$. This proves that T^* induces T in $\mathfrak{o}$, so that $\bar{\mathfrak{o}}$ is a subring and a subspace of $\bar{\mathfrak{o}}^*$. The other contention is part of the following proof of the converse.

Conversely, let $\bar{\mathfrak{o}}$ be a subring and subspace of $\bar{\mathfrak{o}}^*$. Let v be a prime divisor of $k(F)$ whose center on F is G, and let v^* be the usual unique extension of v to $\mathcal{K}(\mathfrak{o})$. R_{v^*} contains some $\mathfrak{O}_{\mathfrak{m}_j}$, and the completion of $\mathfrak{O}_{\mathfrak{m}_j}$ contains some $\bar{\mathfrak{o}}/\mathfrak{r}$; besides, each $\mathfrak{r}$ is related to some $\mathfrak{m}_j$ in this manner. By [13], v^* has an extension $\bar{v}$ to the quotient field of the completion of $\mathfrak{O}_{\mathfrak{m}_j}$, hence to $\mathcal{K}(\bar{\mathfrak{o}}/\mathfrak{r})$, and $\mathcal{C}(\bar{v}/(\bar{\mathfrak{o}}/\mathfrak{r}))=\bar{\mathfrak{m}}/\mathfrak{r}$. Let $\mathfrak{r}_1^*$, $\mathfrak{r}_2^*$, $\cdots$ be the minimal primes of $\mathfrak{r}\bar{\mathfrak{o}}^*$; then $\bar{v}$ has some extension to each $\mathcal{K}(\bar{\mathfrak{o}}^*/\mathfrak{r}_j^*)$, whose center on $\bar{\mathfrak{o}}^*/\mathfrak{r}_j^*$ is $\bar{\mathfrak{m}}^*/\mathfrak{r}_j^*$. On the strength of this fact, and by essentially the same argument as the one employed in step 3 of the proof of Theorem 3.1, we obtain $n(v)$ ins $D[F]e(\bar{\mathfrak{o}}/\mathfrak{r};\ \zeta)$ $=$ord $D^*[G]e(\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*;\ \zeta)$, $n(v)$ having the same meaning which it has in Theorem 3.1. Hence $n(v)$ does not depend on v if and only if $e(\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*;\zeta)e(\bar{\mathfrak{o}}/\mathfrak{r};\zeta)^{-1}$ does not depend on $\mathfrak{r}$; this, because of statement 1 of

Theorem 3.1, and because of the lemma at the end of this proof, completes the proof of assertion 1.

2. The length l of $\mathfrak{P}(G/F)Q(D^*/D)$ is the length of $\mathfrak{m}\mathfrak{o}^*$; we contend that this, in turn, equals the length of $\bar{\mathfrak{m}}\bar{\mathfrak{o}}^*$. In fact, set $\mathfrak{q}=\mathfrak{m}\mathfrak{o}^*$, so that $\bar{\mathfrak{q}}=\bar{\mathfrak{m}}\bar{\mathfrak{o}}^*=\mathfrak{q}\bar{\mathfrak{o}}^*$ is the closure of $\mathfrak{q}$ according to the $\bar{\mathfrak{o}}^*$-topology. Let $\mathfrak{q}=\mathfrak{q}_1\subset\mathfrak{q}_2\subset\cdots\subset\mathfrak{q}_l=\mathfrak{m}^*$ be a maximal chain of primaries between $\mathfrak{q}$ and $\mathfrak{m}^*$, and consider the chain $\bar{\mathfrak{q}}=\bar{\mathfrak{q}}_1\subset\bar{\mathfrak{q}}_2\subset\cdots\subset\bar{\mathfrak{q}}_l=\bar{\mathfrak{m}}^*$ constructed with the closures of the $\mathfrak{q}_j$'s; assume $\bar{\mathfrak{q}}_j\subseteq\mathfrak{Q}\subset\bar{\mathfrak{q}}_{j+1}$, $\mathfrak{Q}$ being a primary; then $\mathfrak{q}_j\subseteq\mathfrak{Q}\cap\mathfrak{o}^*\subset\mathfrak{q}_{j+1}$, hence $\mathfrak{q}_j=\mathfrak{Q}\cap\mathfrak{o}^*$. Let $q\in\mathfrak{o}^*$ be such that $\mathfrak{q}_{j+1}=\mathfrak{q}_j+q\mathfrak{o}^*$; then $\mathfrak{Q}$ consists of elements of the form $aq+q'$, where $a\in\bar{\mathfrak{o}}^*$ and $q'\in\bar{\mathfrak{q}}_j$; let $\mathfrak{a}$ be the set consisting of the elements $a\in\bar{\mathfrak{o}}^*$ such that $aq+q'\in\mathfrak{Q}$ for some (hence for each) $q'\in\bar{\mathfrak{q}}_j$; $\mathfrak{a}$ is an ideal of $\bar{\mathfrak{o}}^*$, and $\mathfrak{Q}=q\mathfrak{a}+\bar{\mathfrak{q}}_j\subseteq q\bar{\mathfrak{m}}^*+\bar{\mathfrak{q}}_j$ since $\mathfrak{a}$ is proper. Now, $q\mathfrak{m}^*\subseteq\mathfrak{q}_j$, hence $q\bar{\mathfrak{m}}^*\subseteq\bar{\mathfrak{q}}_j$, so that $\mathfrak{Q}=\bar{\mathfrak{q}}_j$. This proves that the chain $\{\bar{\mathfrak{q}}_j\}$ is maximal, or that l is also the length of $\bar{\mathfrak{m}}\bar{\mathfrak{o}}^*$.

Now, $[(\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*)\mathcal{K}(\bar{\mathfrak{o}}/\mathfrak{r}):\mathcal{K}(\bar{\mathfrak{o}}/\mathfrak{r})]e(\bar{\mathfrak{o}}/\mathfrak{r};\ \zeta)=e(\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*;\ \zeta)[\bar{\mathfrak{o}}^*/\bar{\mathfrak{m}}^*:\bar{\mathfrak{o}}/\bar{\mathfrak{m}}]$, and therefore $e(D^*/D;\ V, G)^*=e(\bar{\mathfrak{o}}^*;\zeta)e(\bar{\mathfrak{o}};\zeta)^{-1}=e(\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*;\zeta)e(\bar{\mathfrak{o}}/\mathfrak{r};\zeta)^{-1}=[(\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*)\times\mathcal{K}(\bar{\mathfrak{o}}/\mathfrak{r}):\mathcal{K}(\bar{\mathfrak{o}}/\mathfrak{r})][\mathfrak{o}^*/\mathfrak{m}^*:\mathfrak{o}/\mathfrak{m}]^{-1}\leqq\text{length }(\bar{\mathfrak{m}}/\mathfrak{r})(\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*)=\text{length }\bar{\mathfrak{m}}\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*=l$ by Theorem 8 of [4].

3. This is a consequence of the formula $e(D^*/D;\ V, G)^*=[\mathcal{K}(\bar{\mathfrak{o}}^*):\mathcal{K}(\bar{\mathfrak{o}})]\times[\mathfrak{o}^*/\mathfrak{m}^*:\mathfrak{o}/\mathfrak{m}]^{-1}$ and of Theorem 23 of [4].

4. This is a consequence of the results of [8].

In the proof of statement 2 we have used the fact that the condition of $e(S/\mathfrak{r}S;\ \zeta)e(R/\mathfrak{r};\ \zeta)^{-1}$ being independent of $\mathfrak{r}$ is equivalent to the condition that $e(\bar{\mathfrak{o}}^*/\mathfrak{r}\bar{\mathfrak{o}}^*;\ \zeta)e(\bar{\mathfrak{o}}/\mathfrak{r};\ \zeta)^{-1}$ is independent of $\mathfrak{r}$ (notice that $\mathfrak{r}$ has different meanings in the two conditions). Now, this is a consequence of the following result:

LEMMA. *Let R be a geometric domain, $\mathfrak{m}$ its maximal prime, $\bar{R}$ its completion; let x be an indeterminate, and set $\mathfrak{o}=R[x]_{\mathfrak{m}R[x]}$; let $\bar{\mathfrak{o}}$ be the completion of $\mathfrak{o}$. Then there is a 1-1 correspondence $\mathfrak{R}\to\mathfrak{r}=\mathfrak{R}\cap\bar{R}$ between the set of the minimal primes $\mathfrak{R}$ of the zero ideal of $\bar{\mathfrak{o}}$ and the set of the minimal primes $\mathfrak{r}$ of the zero ideal of $\bar{R}$. If, in addition, $\{\zeta\}$ is a set of parameters of R, then $e(\bar{\mathfrak{o}}/\mathfrak{R};\ \zeta)=e(\bar{R}/\mathfrak{r};\ \zeta)$ (after identifying the ζ's with their classes* mod $\mathfrak{R}$).

Proof of the lemma. The set of the $\mathfrak{r}$'s is in 1-1 correspondence with the set of the minimal primes of the extension of $\mathfrak{m}$ to the integral closure of R; this set is in 1-1 correspondence with the set of the minimal primes of the extension of $\mathfrak{m}\mathfrak{o}$ to the integral closure of $\mathfrak{o}$, and this in turn is in 1-1 correspondence with the set of the $\mathfrak{R}$'s. This proves that the correspondence $\mathfrak{R}\to\mathfrak{r}$ is 1-1. Let S' be the integral closure of R, $\mathfrak{p}$ a minimal prime of $\mathfrak{m}S'$, $S=S'_{\mathfrak{p}}$, $\mathfrak{O}=S[x]_{\mathfrak{p}S[x]}$. Let $\bar{S}$, $\bar{\mathfrak{O}}$ be the completions of S, $\mathfrak{O}$, and assume that $\mathfrak{p}$ corresponds to $\mathfrak{r}$. It has been shown in case 2 of the proof of Lemma 2.2 that $\mathcal{K}(\bar{S})=\mathcal{K}(\bar{R}/\mathfrak{r})$, so that also $\mathcal{K}(\bar{\mathfrak{O}})=\mathcal{K}(\bar{\mathfrak{o}}/\mathfrak{R})$. As a consequence, $e(S;\ \zeta)[S/\mathfrak{m}S:R/\mathfrak{m}]=e(\bar{S};\ \zeta)[S/\mathfrak{m}S:R/\mathfrak{m}]=e(\bar{R}/\mathfrak{r};\ \zeta)$, and in like manner

$e(\mathfrak{O}; \zeta)[(S/\mathfrak{m}S)(x):(R/\mathfrak{m})(x)] = e(\bar{\mathfrak{o}}/R; \zeta)$. By Lemma 3.1 we have $e(S; \zeta) = e(\mathfrak{O}; \zeta)$, so that $e(\overline{R}/\mathfrak{r}; \zeta) = e(\bar{\mathfrak{o}}/\mathfrak{R}; \zeta)$, Q.E.D.

We now let k, F, D, V, G have the same meaning as in Theorem 5.3, while D^* is a component of $[D; V, G]$, concerning whose dimension no a priori assumption is made. If $\{x_0, \cdots, x_m\}$ is the h.g.p. of $D[F]$, let $X_0, \cdots, X_m$ be indeterminates, so that $\{X_0, \cdots, X_m\}$ is the h.g.p. of the projective space over $k(F)$ of which $D[F]$ is subvariety. $\{X_0, \cdots, X_m\}$ is also the h.g.p. of the projective space over $k(G)$ of which $D^*[G]$ is subvariety. Let $\{z\}$ be a n.h.g.p. of F for which G is at finite distance, $\{\bar{z}\}$ the corresponding n.h.g.p. of G, $\{\bar{x}\}$ the h.g.p. of $D^*[G]$. If p is the characteristic of k, and if $p\neq 0$, let $\{t_1, t_2, \cdots\}$ be a p-independent basis of $k(G)$ over $k(G)^p$, and consider the derivations $\partial/\partial X_i$, $\partial/\partial t_j$ of $k(X, \bar{z})$ over $k(G)^p$; if $p=0$ or if $k(G)$ is perfect the $\partial/\partial t_j$ simply do not exist, and the $\partial/\partial X_i$ are derivations over $k(G)$. Finally, let $\{f_1(X, z), f_2(X, z), \cdots\}$ be a basis of $\wp(D/k[X, z])$. We shall denote by $J(f(X, \bar{z}); X, t)$ the jacobian matrix of the $f_i(X, \bar{z})$ with respect to the derivations $\partial/\partial X_i$, $\partial/\partial t_i$. The following result is a criterion for multiplicity one expressed for homogeneous x's and nonhomogeneous z's; the easy transformation of the criterion to the cases in which the x's are nonhomogeneous, or the z's are homogeneous is left to the reader.

THEOREM 5.6 (JACOBIAN CRITERION FOR MULTIPLICITY ONE). *Using the previous notation, and assuming G to be simple on F, we have that the two following statements are equivalent*:

1. $\dim D^* = \dim D - \dim F + \dim G$, *and* $e(D^*/D; V, G)^* = 1$;
2. $J(f(X, \bar{z}); X, t)$ *acquires the rank* $(m - \dim D + \dim F)$ *when the* X's *are replaced by the* $\bar{x}$'s.

Proof. If $\dim D^* = \dim D - \dim F + \dim G$ and $e(D^*/D; V, G)^* = 1$, then $\mathfrak{m}R[x]$ has the isolated primary component $\mathfrak{p}' = \wp(D^*/k[x, z])R[x]$; here we have denoted by $\mathfrak{m}$, R respectively $\mathfrak{P}(G/F)$ and $Q(G/F)$. This means that $\mathfrak{m}R[X] + \mathfrak{p}R[X]$ has the isolated primary component $\wp(D^*/k[X, z])R[X]$, after putting $\mathfrak{p} = \wp(D/k[X, z])$. Reducing mod $\mathfrak{m}R[X]$ we obtain that $\{f_1(X, \bar{z}), f_2(X, \bar{z}), \cdots\}$ is the basis of an ideal of which $\mathfrak{P} = \wp(D^*[G]/k(G)[X])$ is an isolated primary component. Therefore there exists a regular set of parameters of $k(G)[X]_{\mathfrak{P}}$ every element of which is a linear combination of the $f_i(X, \bar{z})$'s with coefficients in $k(G)[X]$. But then Theorem 10 of [14] yields the contention expressed in statement 2, since $m - \dim D + \dim F$ is the dimension of $k(G)[X]_{\mathfrak{P}}$.

Conversely, assume $J(f; X, t)$ to have the desired rank for $X = \bar{x}$. Then $\dim D^* = \dim D - \dim F + \dim G$, otherwise J would have rank less than $m - \dim D + \dim F$, independently of whether $\{f\}$ is a basis of $\mathfrak{P}k(G)[X]_{\mathfrak{P}}$ or not. Furthermore, $(m - \dim D + \dim F)$ among the f's form a regular set of parameters of $k(G)[X]_{\mathfrak{P}}$ because of Theorem 10 of [14], and this shows that

$\{f\}$ generates an ideal of which $\mathfrak{P}$ is an isolated primary component. We can then retrace the previous steps, obtaining that $\mathfrak{m}R[x]$ has the isolated primary component $\mathfrak{p}'$. Since R is regular, this shows that $e(D^*/D;\ V,\ G)^* = 1$, Q.E.D.

COROLLARY. *With the same notation as of Theorem* 5.6, *we have that the following three statements are equivalent to each other*:

1. $\dim D^* = \dim D - \dim F + \dim G$, $\text{ins}\ D^*[G] = 1$, *and* $e(D^*/D;\ V,\ G)^* = 1$;

2. $J(f(X, \bar{z});\ X)$ *acquires the rank* $m - \dim D + \dim F$ *when* $\{X\}$ *is replaced by* $\{\bar{x}\}$;

3. $J(f(X, z);\ X)$ *acquires the rank* $m - \dim D + \dim F$ *when* $\{X\}$ *is replaced by* $\{\bar{x}\}$ *and* $\{z\}$ *by* $\{\bar{z}\}$; *in this case* J *denotes the jacobian matrix with respect to the derivations* $\partial/\partial X_i$ *of* $k(F)(X)$ *over* $k(F)$.

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